Budapest University of Technology and Economics
Department of Control and Transport Automation

Analysis of compartmental traffic models

János Polgár

Master's thesis

Göteborg - Budapest, Autumn 2011

Abstract:

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Keywords:

Discrete time compartmental systems, bilinear systems, macroscopic traffic models

Supervisors:

Balázs Kulcsár
Chalmers University of Technology Department of Signals and Systems
SE-412 96, Göteborg, Hörsalsvägen 9-11.
email: kulcsar@chalmers.se

Tamás Luspay
MTA SZTAKI Systems and Control Lab
H-1111, Budapest, Kende u. 13-17.
email: tluspay@sztaki.hu

Co-supervisor:

István Varga
MTA SZTAKI Systems and Control Lab
H-1111, Budapest, Kende u. 13-17.
email: ivarga@sztaki.hu

Acknowledgement

First of all, I would like to express my gratefulness to my supervisor, Balázs Kulcsár. He has given me a lot of help during the development of my thesis, and has permanently been guiding through this way.

I thank Tamás Luspay for his creative ideas respecting the compartmental model.

This thesis could not have come off without the support of Gábor Szederkényi.

Bo Egardt and Claes Breitholtz have provided me an inspiring working environment at the Department of Signals and Systems at Chalmers University of Technology.

To István Varga and Tamás Tettamanti I would like to thank their support during my studies at BUTE.

András Kertész has helped me with the revision of the text.

Contents

1 Introduction

1.1 Traffic models

The system theory has been dealing with the modeling of traffic process for a long time. Several models have been developed to model the behaviour of the traffic flow. These models can be grouped in many ways. Some models are concerned with the urban traffic, and others describe the freeway traffic. The level of modeling varies, as well. In the microscopic models each vehicle is considered independently with its properties concerning the speed, acceleration, lane changes, etc. The interaction between the following vehicles is considered in these models. For instance, Wiedemann's car following model belongs to this group [27].

The macroscopic traffic models consider vehicle groups. The traffic flow is characterized by some overall paramters, such as time mean speed, flow quantity, density, etc. These models have been derived from fluid dynamic models. The first macroscopic traffic flow model was the Lighthill-Whitham-(Richards) model [14, 19], which describes the traffic flow on freeways. The Cell Transmission Model [7, 8] is able to model both the freeway and the urban traffic flow. The Store & Forward model [2, 3, 10] is used in realized control algorithm in more urban location.

Between the microscopic and macroscopic models can be found, as a transition, the mesoscopic models. These models use single vehicles with its properties in continuous time. But they usually apply spatially discretized network. What does the discretized network (and time) mean in case of the traffic flow models?

The traffic flow is a continuous time process. But our control opportunities (most often traffic lights) are able to operate in discrete time. The green times are computed in terms of seconds, thus, the smallest unit which the controller has to use - not a surprise - is the second. To control and model the movement of the vehicles, the continuous time process has to be temporally discretized.

The long freeway should be divided into shorter (often equal long) parts, to segregate the various traffic situations. Each sector is described by a distinct equation. This process is called spatial discretization. In the urban environment, the shorter road links can be considered as a discretized network without further assumptions.

Similarly to a lot of natural processes, the traffic flow can be modeled as a nonnegative system. Roughly, nonnegativity means that the system will give nonnegative response to nonnegative inputs. In case of the traffic network, the positive input is the number of entering vehicles, which cannot be negative. The output of the system can be, for instance, the number of vehicles on a certain link, or the number of outflowing vehicles from a subnetwork. These values are naturally nonnegative. The traffic models have to be able to manage this property. The model equations have to be constructed such a way that the model should not give negative results.

Moreover, the vehicle-conservation law is fulfilled in the traffic flow. This law declares that the vehicles cannot disappear in the network, and vehicles cannot arise in the network without a defined source. (This law is a special form of the mass-conservation law, applied in physical systems.)

1.2 Motivation

The Store & Forward model has been widely examined in the past decade. A main advantage of this model is that it produces a linear system to describe the road traffic in urban areas. The linear systems can be treated well, if we design control, or analyze the properties of the model.

Unfortunately, this model is not able to handle the uncertainties obtained in the modeling procedure [23].

The uncertainties come from our measurement systems, and caused by the modeling method. The installed, and now employed measurement systems (loop detectors and cameras) in road traffic cannot count the number of vehicles without fault. Moreover, there can be links in the traffic network, in which we do not measure any parameters. There can be turning possibilities on the unmeasured part of the network between two measured links, which guide some vehicles out from the considered network. Thus, the amount of the unmeasured vehicles misses from the measured state of the next link. This lack of information can be considered as a fault, as well. The traffic models without any assumption of measurement failure can be considered as so called nominal systems. If we design controller to this nominal model to control the movement of the vehicles by traffic lights, we cannot reach the appointed output (outflow of vehicles) by the computed control inputs (green signals). We have to use uncertain models, in which there are other terms representing the measurement failures. The controls, which are applied to the uncertain model, are called robust controls [28].

It can be a reasonable question if there are any modeling approaches, which can apply the uncertainty in the road traffic measurement, and can be constructed without a lot of constraints and additional assumptions. Moreover, it is expected that in the system matrices there can be found some indication to the traffic network structure. The aim of this thesis is that the so called compartmental systems are whether suitable to model the urban traffic with the above considerations. Hence, analysis of compartmental behaviour for macroscopic and urban traffic flow modeling is considered. One of our intention is to adress the control oriented modeling question of such a traffic flow description rather than focus on pure control synthesis techniques.

The compartmental models are widely used in the modeling chemical processes, or biological processes (see for instance, [6, 17, 21]), mostly. These models are constituted in continuous time. Since our control techniques in the road traffic are not able to handle the continuous time (the data collection, as well as the control can be realized between discrete time steps), we need discrete time models and controllers. Naturally, discrete time compartmental models can be used, as well. The compartmental models have two important properties, which are definitely required to use them as traffic flow model. First, the mass-conservation law is always held, and second, they are nonnegative, as well. As key feature, the movement of the vehicles can be considered as the mass flow.

This thesis is organized as follows. The second section introduces the required notations, and provides some definitions and theorems from the field of linear algebra, and the system theory. Section 3 gives an overview from the macroscopic traffic flow models Store & Forward model, and the Cell Transmission Model. In the fourth section, the exposition of the compartmental traffic model, including the properties of the matrix representations is presented. A small simulation example is presented in this section, as well. The fifth section deals with the analysis of the compartmental traffic model. The stability in Lyapunov sense, the local controllability, and the sensitivity of the system matrix have been examined, and have been illustrated by examples. Section 6 marks the further research directions out, and Section 7 contains some concluding remarks about the results of this thesis.

2 Notations and preliminaries

The applied notations are reviewed in this section. Some definitions are given from the field of linear algebra and system theory, as well.

Abbrevations

CTM	: *C*ell *T*ransmission *M*odel
LTI	: *L*inear *T*ime *I*nvariant (system)
SF	: *S*tore & *F*orward (model)

Mathematical notations

$\mathbb{R}^n$	: n dimensional field of real numbers
$\mathbb{R}^{m \times n}$	: $m \times n$ dimensional field of real numbers
$\overline{\mathbb{R}^n_+}$	: nonnegative n dimensional field of real numbers ($x \geq 0$)
$\overline{\mathbb{Z}_+}$	: field of nonnegative integers
$\mathbb{N}$	: field of natural numbers
$\mathbf{x}$, $\mathbf{y}$	: vectors denoted by bold face letters
$\mathbf{x} \geq\geq 0$	: all entries of $\mathbf{x}$ are nonnegative
$\mathbf{A} \in \mathbb{R}^{m \times n}$	: real valued matrices with n row and m column, denoted by bold face capital letters
$\mathbf{A}^T$	: the transpose of $\mathbf{A}$
$\mathbf{A}^{-1}$	: the inverse of $\mathbf{A}$
$\lvert\mathbf{A}\rvert$	: the determinant of $\mathbf{A}$
λ	: eigenvalue of a matrix
$\mathbf{v}$	: eigenvector of a matrix

Notations of variables

Variables related to the SF model

$q_i(k)$	: number of entering vehicles into the i-th link at time step k
$h_i(k)$	: number of leaving vehicles from the i-th link at time step k
$d_i(k)$	: number of entering vehicles from environment into the i-th link at time step k (demand input)
$o_i(k)$	: number of leaving vehicles to environment from the i-th link at time step k
$x_i(k)$	: number of vehicles in the i-th link at time step k
α_{ij}	: turning rate from the j-th link to the i-th link
s_i	: leaving capacity in the i-th link
$u_i(k)$	: green time of the i-th link at time step k
T_c	: cycle time of the signal plan

Variables related to the compartmental traffic model

$q_i(k)$	: number of entering vehicles into the i-th compartment at time step k
$h_i(k)$	: number of leaving vehicles from the i-th compartment at time step k
$d_i(k)$	: number of entering vehicles from the environment into the i-th compartment at time step k (demand input)
$o_i(k)$	: number of the leaving vehicles to the environment from the i-th compartment

	at time step k
$x_i(k)$	: number of vehicles in the i-th compartment at time step k
$\beta_{ij}(k)$	: leaving flow rate from the j-th compartment to the i-th compartment at time step k
p	: number of permitted turns in the entire compartmental network
$u_i(k)$	: green time of the i-th compartment at time step k
$r_{i,j}$	: transmission throughput rate from the compartment j to the compartment i
$r'_{i,j}$	: maximal transmission from the compartment j to the compartment i
x_i^{max}	: maximal capacity of the i-th compartment
T_s	: sample time

Definitions

The presented compartmental systems are nonnegative systems, therefore they contain positive matrices. Hence, the definition of the nonnegative matrix is provided.

Definition 2.1 (Positive matrix)
The matrix $\mathbf{N}$ *is called positive, if all its entries are nonnegative, and at least one is positive (to avoid the trivial case of all-zero matrix) [4].*

In model analysis the eigenvectors and eigenvalues are required many times. Some parameters can be introduced only with the help of these notions. Moreover, the eigenvalues have physical sense in the compartmental traffic system.

Definition 2.2 (Eigenvectors and eigenvalues of a matrix)

1. *Let us consider the following equation with the real matrix* $\mathbf{N}$,

$$\mathbf{N}\mathbf{v} = \lambda\mathbf{v}. \tag{2.1}$$

   The nonzero vector $\mathbf{v}$ *is said to be the right eigenvector of the matrix* $\mathbf{N}$, λ *is the eigenvalue of the matrix* $\mathbf{N}$. *The right eigenvectors are column vectors.*

2. *Let us consider the equation with the real matrix* $\mathbf{N}$,

$$\mathbf{y}^T\mathbf{N} = \lambda\mathbf{y}^T. \tag{2.2}$$

   The nonzero vector $\mathbf{y}^T$ *is said to be the left eigenvector of the matrix* $\mathbf{N}$, λ *is the eigenvalue of the matrix* $\mathbf{N}$. *The left eigenvectors are row vectors. (To this fact refers the superscript* ^{T}*.) [20]*

The stability analysis requires positive-definite matrices in certain cases.

Definition 2.3 (Positive-definite real matrix)
The $n \times n$ *real matrix* $\mathbf{N} \in \mathbb{R}^{n\times n}$ *is positive-definite, if* $\mathbf{x}^T\mathbf{N}\mathbf{x} > 0$ *is satisfied for all non-zero vectors* $\mathbf{x}$ *with real entries* ($\mathbf{x} \in \mathbb{R}^n$).

With other words, the $n \times n$ *real matrix* $\mathbf{N} \in \mathbb{R}^{n\times n}$ *is positive-definite, if all its eigenvalues are positive* ($\lambda_i > 0 \quad i = 1, \cdots, n$)*, where* λ *denotes the eigenvalues of* $\mathbf{N}$ *[20].*

The properties of the matrix representation of the compartmental traffic systems are examined. The negative-semidefinitness appears among these properties.

Definition 2.4 (Negative-semidefinite real matrix)
The $n \times n$ real matrix $\mathbf{N} \in \mathbb{R}^{n\times n}$ is negative-semidefinite, if $\mathbf{x}^T\mathbf{N}\mathbf{x} \leq 0$ is satisfied for all non-zero vectors $\mathbf{x}$ with real entries ($\mathbf{x} \in \mathbb{R}^n$).

With other words, the $n\times n$ real matrix $\mathbf{N} \in \mathbb{R}^{n\times n}$ is negative-semidefinite , if all its eigenvalues are nonpositive ($\lambda_i \leq 0 \quad i = 1, \cdots, n$), where λ denotes the eigenvalue of $\mathbf{N}$ [20].

We obviously need the definition of compartmental marices in the analysis of the compartmental systems.

Definition 2.5 (Compartmental matrix)
The $n \times n$ matrix $\mathbf{N} = [n_{ij}]$ is said to be compartmental if the followings are satisfied [24]:

$$1. \quad n_{ij} \geq 0, \quad \text{for all } i, j \in \overline{\mathbb{Z}_+};$$
$$2. \quad \sum_{i=1}^{n} n_{ij} \leq 1, \quad \text{for all } j \in \overline{\mathbb{Z}_+}.$$

The diagonalizability of the matrix representation is examined in this thesis, thus some definitions are required from this field.

Definition 2.6 (Eigenvalue decomposition (EVD))
The $n \times n$ matrix $\mathbf{N}$ can be diagonalizable, if there exists a diagonal matrix $\mathbf{D}$, and an invertible matrix $\mathbf{T}$, such that $\mathbf{D} = \mathbf{T}^{-1}\mathbf{N}\mathbf{T}$. The matrix $\mathbf{D}$ contains the eigenvalues of the matrix $\mathbf{N}$. [26]. (The eigenvalue decomposition is also called as diagonalizing of a matrix.)

Theorem 2.7 (Necessary and sufficient condition for existing the EVD)
The $n \times n$ matrix $\mathbf{N}$ can be diagonalizable if and only if it has n linearly independent eigenvectors [26]. The proof can be found in [26].

The modal matrix plays an important role in the eigenvalue decomposition.

Definition 2.8 (Modal matrix)
The columns of the modal matrix of the $n \times n$ matrix $\mathbf{N}$ is constituted by the eigenvectors of the matrix $\mathbf{N}$. At eigenvalue decomposition (See Definition 2.6.), the modal matrix fill the part of the transformation matrix $\mathbf{T}$ [20].

Some examined parameters are defined by matrix norms. The definition of the matrix norm need the notion of the singular value.

Definition 2.9 (Singular value of a real matrix)
If the $n \times n$ matrix $\mathbf{N}$ is a real valued, and has rank r, and σ_i^2 denotes the eigenvalues of the positive semidefinite matrix $\mathbf{N}^T\mathbf{N}$, then the numbers:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r = \sigma_{r+1} = \cdots = \sigma_n$$

are called as the singular values of the matrix $\mathbf{N}$ [20].

Definition 2.10 (p norm of a matrix)
The p-norm of a matrix $\mathbf{N} \in \mathbb{R}^{q\times r}$ is definied by the following expression.

$$||\mathbf{A}|| = \max_{\mathbf{x}\neq\mathbf{0}} \frac{||\mathbf{A}\mathbf{x}||_p}{||\mathbf{x}||_p}, \tag{2.3}$$

where $p \in \{1, 2, \cdots, \infty\}$. There are formulas for some p to compute the corresponding p-norm [20].

- $||\mathbf{A}||_1 = \max_{1\leq j\leq r} \sum_{i=1}^{q} |n_{i,j}|$,
- $||\mathbf{A}||_2 = \sigma_1$, *i.e. the greatest singular value of matrix* $\mathbf{N}$,
- $||\mathbf{A}||_\infty = \max_{1\leq i\leq q} \sum_{j=1}^{r} |n_{i,j}|$.

The compartmental systems are nonnegative systems, therefore we have to define this class of systems.

Definition 2.11 (Nonnegative system)
The discrete time linear dynamical system given by

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \quad \mathbf{x}(0) = \mathbf{x}_0, k \in \mathbb{N}, \mathbf{x} \in \mathbb{R}^n \tag{2.4}$$

is nonnegative if for every $\mathbf{x}(0) \in \overline{\mathbb{R}_+^n}$ *and* $\mathbf{u}(k) \geq\geq 0$, $k \in \mathbb{N}$, *the solution* $\mathbf{x}(k)$, $k \in \mathbb{N}$, *to (2.4) is nonnegative [5].*

Proposition 2.12
The linear dynamical system given by (2.4) is nonnegative if and only if $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, *and* $\mathbf{x}(0)$ *is nonnegative. [5]*

Our compartmental traffic system has a bilinear form.

Definition 2.13 (Discrete time bilinear system)
A discrete time bilinear system is defined by the difference equation

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{x}(k))\mathbf{u}(k) + \mathbf{C}\mathbf{u}(k) \tag{2.5}$$

An equivalent formulation of the equation (2.5) can be the following equation, if the system has only one input:

$$\mathbf{x}(k+1) = (\tilde{\mathbf{A}} + \mathbf{u}(k)\tilde{\mathbf{B}})\mathbf{x}(k) + \tilde{\mathbf{C}}\mathbf{u}(k), \tag{2.6}$$

where $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$ *are real constant matrices in appropriate dimensions [12].*

The model analysis extends to the field of stability analysis in Lyapunov-sense.

Definition 2.14 (Equilibrium point of a discrete time system)
The point x_e *of the dynamical system given by* $\mathbf{x}(k+1) = f(\mathbf{x}(k); 0; k)$ *is said to be an equilibrium point from the time step* k_0, *if [9]*

$$\mathbf{x}(k) = \mathbf{x}_e \quad \forall k \geq k_0. \tag{2.7}$$

Definition 2.15 (Lyapunov stability for a solution of a nonnegative dynamical system)
The equilibium solution $\mathbf{x}(k) \equiv \mathbf{x}_e$ *of the nonnegative system given by*

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad k \in \mathbb{N} \tag{2.8}$$

is Lyapunov stable, if for every $\varepsilon > 0$ *there exist* $\delta = \delta(\varepsilon) > 0$ *such that if* $\mathbf{x}_0 \in B_\delta(\mathbf{x}_e) \cap \overline{\mathbb{R}_+^n}$, *then* $\mathbf{x}(k) \in B_\varepsilon \cap \overline{\mathbb{R}_+^n}$, $k \in \mathbb{N}$, *where* $B_\delta(\mathbf{x}_e)$ *is a hypersphere around* $\mathbf{x}_e$ *with radius* δ.
The equilibrium solution $\mathbf{x}(k) \equiv \mathbf{x}_e$ *of the nonnegative dynamical system (2.8) is semistable, if it is Lyapunov stable, and there exists* $\delta > 0$ *such that if* $\mathbf{x}_0 \in B_\delta(\mathbf{x}_e) \cap \overline{\mathbb{R}_+^n}$, *then* $\lim_{k\to\infty} \mathbf{x}(k)$ *exists.*
The equilibrium solution $\mathbf{x}(k) \equiv \mathbf{x}_e$ *of the nonnegative dynamical system (2.8) is asymptotically stable if it is Lyapunov stable and there exists* $\delta > 0$ *such that if* $\mathbf{x}_0 \in B_\delta(\mathbf{x}_e) \cap \overline{\mathbb{R}_+^n}$, *then* $\lim_{x\to\infty} \mathbf{x}(k) = \mathbf{x}_e$ *[13].*

The controllability of the systems is strongly connected to the reachability property.

Definition 2.16 (Reachability of nonlinear systems)

1. *For the system*

$$\mathbf{x}(k+1) = f(k, \mathbf{x}(k), \mathbf{u}(k)) \quad (k \in \mathbb{Z}_+) \tag{2.9}$$

the state $\mathbf{x} \in (\mathbb{R}^n \times \mathbb{Z})$ *is reachable (or N-step reachable), if there exist a control sequence* U *such that the state* 0 *is transferred to* $\mathbf{x}$ *under the action of* U $\left(0 \quad \overset{U}{\mapsto} \quad \mathbf{x}\right)$.

2. *If for all* $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}$ *is reachable (or N-step reachable), then the system (2.9) is completely reachable at all times [25].*

Definition 2.17 (Reachability of LTI systems)

The discrete time LTI system given by (2.4) is reachable, if and only if the discrete time reachability matrix, denoted by $\mathbf{W}_r$,

$$\mathbf{W}_r = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \tag{2.10}$$

has full rank, i.e. $\text{rank}(\mathbf{W}_r) = n$.

Remark 2.18

If $\mathbf{A}$ *is reversible, i.e* $\exists \mathbf{A}^{-1}$ *then discrete time reachability coincidences with controllability.*

3 Macroscopic urban traffic models

Some theoretical assupmtions of the presented two macroscopic urban traffic models are considered during the development of the compartmental traffic model. Therefore a short outline is presented from the Store & Forward model, and the Cell Transmission Model.

3.1 The Store & Forward model

In this subsection a short summary of the Store & Forward (SF) model [2, 3, 10] is given. This approach can be used to model the dynamics of the urban traffic flow. While the real vehicles are moving in the network, actually they travel from a traffic light to another. During this movement they do turns in some intersections. This moving-stopping motion appears in the SF model. As state variables of the SF model are considered the queue lengths in the road links. This approach applies the so called turning rates (denoted by α). This parameter describes how many vehicles turn into the several directions after passing the stop line. The SF model depends on the knowledge of all turning rates in each junction. The outflow of the link, and the green time is connected by a leaving capacity (denoted by s). It is assumed that all vehicles can leave a link and therefore $s_i u_i(k)$ reflects the real outflow.

The notations of the SF approach can be seen in Figure 1.

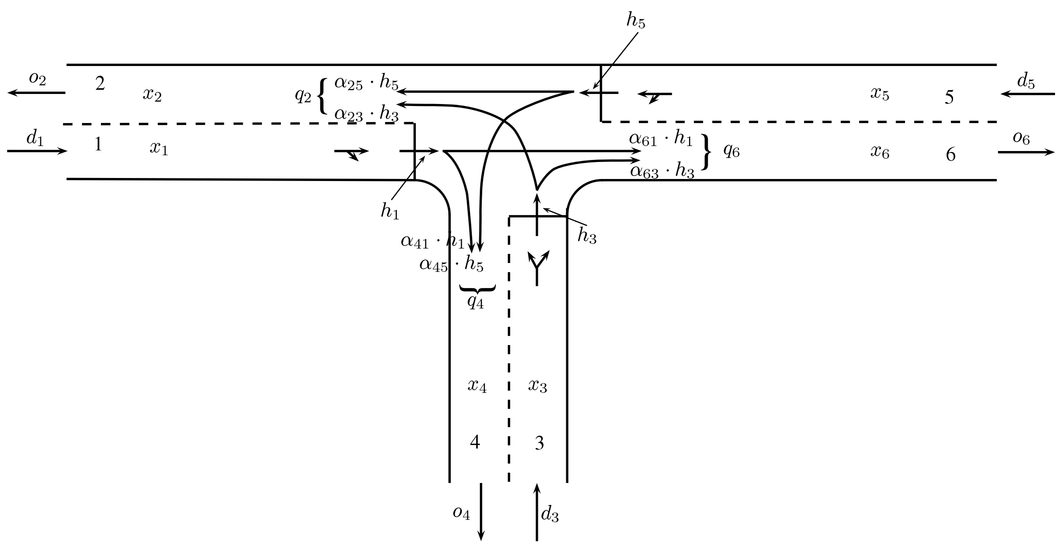

Figure 1: The notations of the SF model

The model equation describing the dynamics of the system is given by the following,

$$x_i(k+1) = x_i(k) + q_i(k) - h_i(k) + d_i(k) - o_i(k). \tag{3.1}$$

The number of entering vehicles into a link, the number of leaving vehicles from a link can be written by the following detailed formulas:

$$q_i(k) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{ij} \cdot h_j(k), \tag{3.2}$$

$$h_i(k) = s_i u_i(k), \tag{3.3}$$

$$o_i(k) = \alpha_{0i} x_i(k). \tag{3.4}$$

Considering the above relationships we can write,

$$x_i(k+1) = x_i(k) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha_{ij} s_j u_j(k) - s_i u_i(k) + d_i(k) - \alpha_{0i} x_i(k). \tag{3.5}$$

The matrix equation form of the open loop system is then given by,

$$\begin{bmatrix} x_1(k+1) \\ \vdots \\ x_i(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 1-\alpha_{01} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1-\alpha_{0i} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1-\alpha_{0n} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_i(k) \\ \vdots \\ x_n(k) \end{bmatrix} +$$

$$+ \begin{bmatrix} -s_1 & \cdots & \alpha_{1i}s_i & \cdots & \alpha_{1n}s_n \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \alpha_{i1}s_1 & \cdots & -s_i & \cdots & \alpha_{in}s_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}s_1 & \cdots & \alpha_{ni}s_i & \cdots & -s_n \end{bmatrix} \begin{bmatrix} u_1(k) \\ \vdots \\ u_i(k) \\ \vdots \\ u_n(k) \end{bmatrix} + \begin{bmatrix} d_1(k) \\ \vdots \\ d_i(k) \\ \vdots \\ d_n(k) \end{bmatrix}. \tag{3.6}$$

The control algorithms always contain feedback, but they differ in the way of the carry out. Some methods use output feedback, others employ the method of state feedback. In urban traffic applications it is more frequent that the output of the system is not defined explicitly. The control algorithm minimizes one (or more) parameters, but these amounts are derived, they cannot be measured in the real network directly. Therefore, the state feedback method is applied in traffic systems. Naturally, the physical content of the state changes with the traffic models. In the SF approach, the queue lengths mean the ground of the control. Thus, we apply state feedback in control appoaches using the SF model [2].

Choose the control input as it was a linear combination of measurable states $x_i(k)$ (linear control policy is applied), and therefore the equation $u_i = k_i x_i$ is held.

Substituting the previous equation into equation (3.5) can be obtained the following formula,

$$x_i(k+1) = x_i(k) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha_{ij} s_j k_j x_j(k) - s_i k_i x_i(k) + d_i(k) - \alpha_{0i} x_i(k). \tag{3.7}$$

The closed loop system is described by the matrix equation,

$$\begin{bmatrix} x_1(k+1) \\ \vdots \\ x_i(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 1-\alpha_{01}-s_1k_1 & \cdots & \alpha_{1i}s_ik_i & \cdots & \alpha_{1n}s_nk_n \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \alpha_{i1}s_1k_1 & \cdots & 1-\alpha_{0i}-s_ik_i & \cdots & \alpha_{in}s_nk_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}s_1k_1 & \cdots & \alpha_{ni}s_ik_i & \cdots & 1-\alpha_{0n}-s_nk_n \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_i(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} d_1(k) \\ \vdots \\ d_i(k) \\ \vdots \\ d_n(k) \end{bmatrix}. \tag{3.8}$$

Note that $s_i u_i(k)$ does not reflect the exact number of crossing vehicles in reality, therefore the vehicle conservation law is not strictly held. Only an expected (nominal) averaged green time transfer rate (s_i) can be associated with the SF description, which, however, might be applicable. Definitely, the exactness of SF model requires the measurement of s_i. Available control design techniques [3] use this concept for constrained control solutions, and have been concluded to work accurately enough in real cases.

3.2 The Cell Transmission Model

This subsection gives a very short overview of Daganzo's Cell Transmission Model (CTM), which was first developed to describe the traffic flow on freeways [7]. Later it has been used for urban traffic modeling [8].

In this approach a road link is divided into cells. The length of a cell is equal to the distance that a vehicle can achieve in free flow conditions. This can be considered as spatial discretization. Thus, the maximal number of vehicles ($N_i(k)$) in the i-th cell at time step k can be described by the following equation:

$$N_i(k) = l_i \rho_{max}, \tag{3.9}$$

where l_i is the length of the i-th cell, ρ_{max} is the 'jam density'[1]. Another predefinied constant is the maximal capacity of cell transmission, i.e. how many vehicles can flow from the i-th cell to the $i+1$-th between the time step k to $k+1$. This amount is denoted by $Q_i(k)$.

The CTM uses directed graphs (detailed in subsection 4.1) for network representation, as well. The cells are considered as nodes, and the turning possibilities mean the arcs.

Let us denote $x_i(k)$ as the number of vehicles in the i-th cell at time step k. The dynamical equation of the cell can be written as,

$$x_i(k+1) = x_i(k) + y_i(k) - y_{i+1}(k). \tag{3.10}$$

The term $y_i(k)$ denotes the inflow to the i-th cell at time step k.

Daganzo has introduced restrictive conditions for the inflows to hold the vehicle-conservation law in the model under the form:

$$y_i(k) = \min\{x_{i-1}(k), \quad Q_i(k), \quad N_i(k) - x_i(k)\}. \tag{3.11}$$

The last term in this condition means the empty space in the target cell (i) at time step k. These amounts describe the entire flow process. The first concerns the source cell, the second the transmission, and the third term the target cell.

The first cell in each direction models the environment of the modeled network. They provide vehicles for the network. It is assumed that these first cells has infinite capacity in terms of the number of vehicles. The last cell in each branch denotes the environment again. These cells behave as absorbing cells. Similarly to the first cells, the last cells have inifite capacity. Another assumption is that the entire network (except for the first cells) is empty at the first time step.

For instance, the CTM model of a T-shape intersection can be seen in Figure 2. The cells related to the environment of the network are depicted by double framed rectangles, and have the notation $e_1 \cdots e_6$.

The network topology can be aggregated by two basic network types. In *merges* there is one cell, which receives vehicles from two or more cells. In Figure 2 cell 9 can be seen, which has two inputs: cells 6 and 2. The *diverge* is the opposite of the merge. In this structure one cell sends vehicles to two or more cells. For instance, cell 2 behaves as a diverge cell. As it can be seen in the figure, there are only 1:1, 1:n, and n:1 connections. The abbrevations denote how many turning possibilities start from the sending cell of a connection, and how many connections arrive at the target cell of a connection.

Daganzo has been restricted the allowed network part structures: the n:n connections are not permitted. This is the reason for the lack of the left turning possibilities in the example network

[1] Jam density is the maximal value of parameter density in one cell.

in Figure 2. In that case special rules would be required to divide the outflow and the inflow into the appropriate parts. By including an artifical cell this problem can be eliminated, as it can be seen in [8]. The inflows and outflows of the elements of merges and diverges are based on (3.11). The detailed method, and the formulas can be found in [8].

The model initially assumes that the turning proportions are known in each junction. Turning proportion denotes the same notion, as the turning rates in the SF model. The CTM applies the FIFO principle. If we would like to distinguish the lanes depending on the further direction of movement, we have to define detached cell for each direction. By this step we eliminate the possibility of the existence of n:n connections.

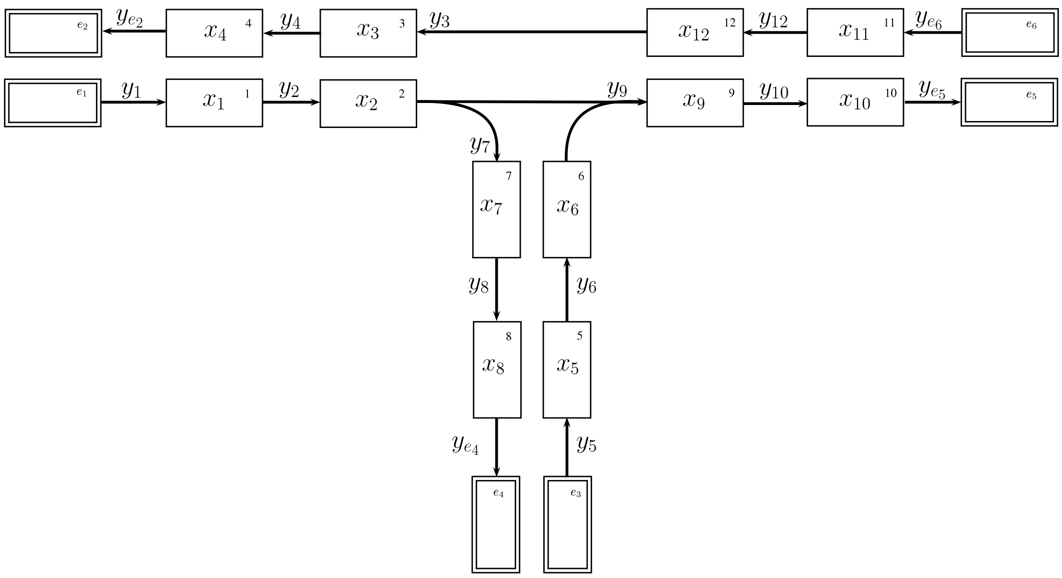

Figure 2: Example network for CTM model

4 The compartmental traffic model

The aim of this chapter is to present the model equations of the compartmental traffic system. Moreover, the properties of the system matrices are discussed here, as well.

4.1 Basics of compartmental systems

Compartmental systems describe the mass transfer between elements of an engineering network. Generally, such elements of a network are represented by compartments in the model, which can contain mass. For instance, tubes, tanks can be taken from the practical engineering fields. If we would like to model the traffic process as a compartmental system, the road links can be considered as compartments, and the nodes (more exactly the turning possibilites) as connections between the compartments.

Conventionally road networks are modeled by a directed graph. (In directed graphs the connections have only one direction, i.e. a bilateral relation is decribed by two connections.) The compartmental approach applies this method, as well. Moreover, each compartment represents one link with only one direction, i.e. a road link with two opposite directional lanes can be represented by two compartments. From the previous two statements it follows that there exists only one connection between two compartments in such model. Another important restriction is that turning from a link to itself is not permitted.

Figure 3 is given to demonstrate the modeling procedure. In the first picture an intersection[2] can be seen, where the turnings from a link towards all possible directions to all directions are permitted. The road links, which will be modeled as compartments, are highlighted by gray rectangles. The turning possibilities mean the connections between the compartments. In the second picture the compartmental model of this intersection can be seen.

As it can be seen in the figure, the road links with more lanes are considered as one compartment. Thus, the multiple lane sectors of the links are not included in the model. If we would consider these, we would have to split the links into two parts: to a free flow part, and a queueing part (as it was done in [15]). To perform this, the information of 'turning rates' are required. These values tell us, the proportion that the vehicles choose a lane before the stop line, when they arrive from the free flow part. For the knowledge of the proportions we would have to measure at more than one sections of the link. These turning rates cannot be controlled, since there is no traffic light before the queueing section.

The model construction without distingushed lanes is appropriate for the model analysis, where we do not need any exact values about the number of turning vehicles. If we have to distinguish the turning directions from a compartment, we can apply an order rule. For instance, it is assumed that the vehicles 'stand' in a compartment in such a queue, where each vehicle is followed by another one that would like to turn to the other direction, than the previous.

In the compartmental traffic model we control the outflow of the compartments, analogously to the real traffic, where the traffic light is placed at the end of the link. The movement of the mass (i.e. the vehicles) in the network is ensured by a single variable called *leaving flow rate*, which is discussed in the following subsection.

Note that subscript 0 will denote the environment in the sequel.

[2]Kapellplatsen in Göteborg, Sweden

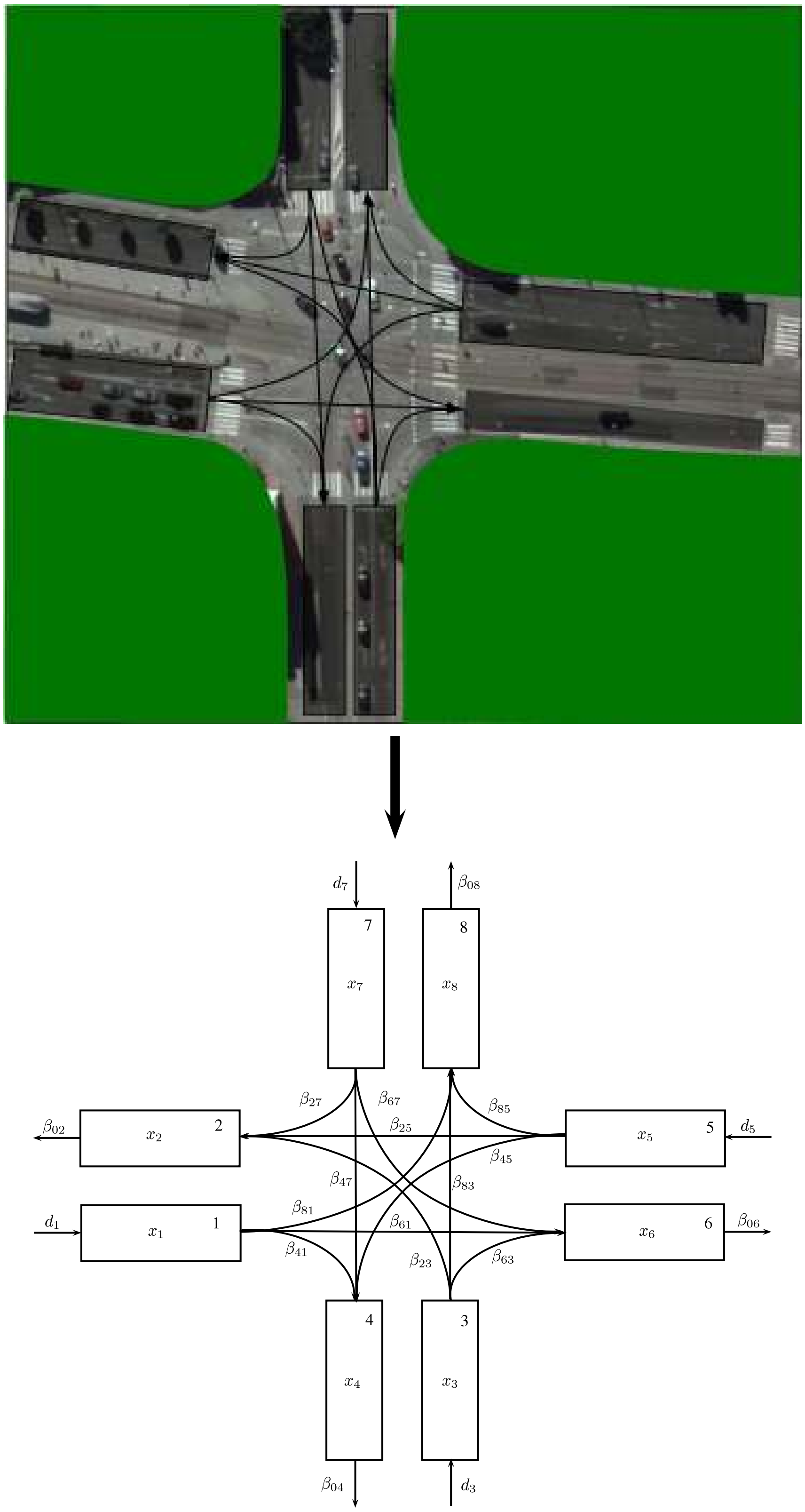

Figure 3: Compartmental model of a real intersection

4.2 Leaving flow rate

This subsection introduces the notion of the leaving flow rate, which is the most important parameter of the compartmental traffic model.

The leaving flow rate combines the notion of the commonly used turning rate and the number of leaving vehicles from a compartment.

Definition 4.1 (Leaving flow rate)
Let $\beta_{ji}(k)$ denote the rate of the leaving vehicles from i-th compartment to the j-th compartment at time step k, i.e.

$$\beta_{ji}(k) = \frac{h_{ji}(k)}{x_i(k)}, \tag{4.1}$$

where $h_{ji}(k)$ denotes the number of leaving vehicles from the i-th compartment to the j-th at time step k, and $x_i(k)$ denotes the number of all vehicles in compartment i at time step k.

For parameter β_{ji} the following constraints have to be satisfied:

$$0 \le \beta_{ji}(k) \le 1, \tag{4.2}$$

$$0 \le \sum_{\substack{l=0 \\ l \ne i}}^{n} \beta_{li}(k) \le 1. \tag{4.3}$$

The ≤ 1 part in the constraint (4.2) ensures that none of the outflows at time step $k+1$ are greater than the amount of the vehicles given in a compartment at time step k consequently, and negative outflow cannot be obtained. The $0 \le$ part follows from the directed property of the compartmental network graph, and the modeling method. The value 0 as a lower bound ensures the positivity of the system.

The constraint (4.3) describes that the sum of the outflows at time step $k+1$ cannot be greater than the amount of vehicles present in the considered compartment at time step k. Equalities are allowed i.e. $\beta_{ji}(k)$ can be both 0 or 1. If there are no vehicles in a compartment only zero number can leave it, in this special case $\beta_{ji}(k) = 0$, $\forall j$. If all vehicles in a compartment leave it during a green signal, then $\beta_{ji}(k) = 1$ or $\sum_{\substack{l=1 \\ l \ne i}}^{n} \beta_{li}(k) = 1$ is taken, depending the number of outflows from the considered compartment.

The leaving flow rate is very similar to the turning rate used in the SF approach (see Section 3.1). But there are two theoretical differences between the notion of turning rates and leaving flow rates. On one hand, it is important to state that the amount $\sum_{\substack{l=1 \\ l \ne i}}^{n} \beta_{li}(k)$ does not have to be equal to 1, unlike in [2, 3, 10]. On the other hand, the parameter β is based on the total amount of vehicles in a compartment instead of just representing the rates of the leaving vehicles, as in SF model.

4.3 Model equations

We formulate the model equations of the compartmental traffic model in this subsection, which use the presented leaving flow rate.

In the sequel we rephrase the previously obtained dynamical equation as,

$$x_i(k+1) = x_i(k) + q_i(k) - h_i(k) + d_i(k) - o_i(k), \tag{4.4}$$

where the terms can be expressed as follows,

$$q_i(k) = \sum_{\substack{j=1\\ j\neq i}}^{n} \beta_{ij}(k)x_j(k), \tag{4.5}$$

$$h_i(k) = \sum_{\substack{l=1\\ l\neq i}}^{n} \beta_{li}(k)x_i(k), \tag{4.6}$$

$$o_i(k) = \beta_{0i}(k)x_i(k). \tag{4.7}$$

Substituting (4.5), (4.6) and (4.7) into (4.4) we can obtain:

$$x_i(k+1) = x_i(k) + \sum_{\substack{j=1\\ j\neq i}}^{n} \beta_{ij}(k)x_j(k) - \sum_{\substack{l=1\\ l\neq i}}^{n} \beta_{li}(k)x_i(k) - \beta_{0i}(k)x_i(k) + d_i(k). \tag{4.8}$$

We can state that our compartmental traffic system has a bilinear structure (Definition 2.13). The obtained dynamical equation is very similar to a system presented in [17]. But the compartmental models in the field of biology are in continuous time domain, and contain a term, which is constructed by a coefficient matrix and a vector of the control inputs. The state in the compartmental traffic system does not depend on the pure control input.

We can compose a matrix equation from the above formula not only one way. Both the state variables ($x_i(k)$), and the control inputs ($\beta_{ji}(k)$) can be factorized out. In part 4.3.1 the control inputs are stacked into a distinct vector. After the matrix equations, the properties of the system matrix are discussed. In part 4.3.2 the state variables are factorized out, and discussed.

4.3.1 Matrix equation form 1

This subsection contains such a matrix representation, where the control inputs are factorized out to a distinct vector.

In an n-dimensional compartmental system the matrix equality (4.9) is valid. In this equation the parameters β with the same indices (i.e. β_{ii}) are also indicated. In practice the reverse turning from i-th compartment to the same compartment is however not permitted. This condition is therefore considered as full 0 columns in matrix $\mathbf{B}$. The full 0 column is located at the i-th column of each part matrix $\mathbf{B}_i$. To simplify notations we decompose $\mathbf{B}$. In virtue of shorthand notations (4.9) can be rewritten to (4.10),

$$\begin{bmatrix} x_1(k+1)\\ \vdots \\ x_i(k+1)\\ \vdots\\ x_n(k+1)\end{bmatrix} = \begin{bmatrix}1&\cdots&0&\cdots&0\\ \vdots&\ddots&\vdots&\vdots&\vdots\\ 0&\cdots&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&\cdots&0&\cdots&1\end{bmatrix}\begin{bmatrix}x_1(k)\\ \vdots\\ x_i(k)\\ \vdots\\ x_n(k)\end{bmatrix} + \begin{bmatrix}d_1(k)\\ \vdots\\ d_i(k)\\ \vdots\\ d_n(k)\end{bmatrix} +$$

$$+ \left[\underbrace{\begin{matrix}0&\cdots&x_i(k)&\cdots&x_n(k)\\ \vdots&\ddots&\vdots&\vdots&\vdots\\ 0&\cdots&-x_i(k)&\cdots&0\\ \vdots&\ddots&\vdots&\vdots&\vdots\\ 0&\cdots&0&\cdots&-x_n(k)\end{matrix}}_{\mathbf{B}_1}\begin{matrix}\cdots\\ \cdots\\ \cdots\\ \cdots\\ \cdots\end{matrix}\underbrace{\begin{matrix}-x_1(k)&\cdots&0&\cdots&0\\ \vdots&\ddots&\vdots&\vdots&\vdots\\ 0&\cdots&-x_i(k)&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ x_1(k)&\cdots&x_i(k)&\cdots&0\end{matrix}}_{\mathbf{B}_n}\underbrace{\begin{matrix}-x_1(k)&\cdots&0&\cdots&0\\ \vdots&\ddots&\vdots&\vdots&\vdots\\ 0&\cdots&-x_i(k)&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&\cdots&0&\cdots&-x_n(k)\end{matrix}}_{\mathbf{B}_0}\right]\begin{bmatrix}\beta_{11}(k)\\ \vdots\\ \beta_{1i}(k)\\ \vdots\\ \beta_{1n}(k)\\ \vdots\\ \beta_{n1}(k)\\ \vdots\\ \beta_{ni}(k)\\ \vdots\\ \beta_{nn}(k)\\ \beta_{01}(k)\\ \vdots\\ \beta_{0i}(k)\\ \vdots\\ \beta_{0n}(k)\end{bmatrix} \tag{4.9}$$

$$\begin{bmatrix} x_1(k+1)\\ \vdots \\ x_i(k+1)\\ \vdots\\ x_n(k+1)\end{bmatrix} = \begin{bmatrix}1&\cdots&0&\cdots&0\\ \vdots&\ddots&\vdots&\vdots&\vdots\\ 0&\cdots&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&\cdots&0&\cdots&1\end{bmatrix}\begin{bmatrix}x_1(k)\\ \vdots\\ x_i(k)\\ \vdots\\ x_n(k)\end{bmatrix} + \begin{bmatrix}\mathbf{B_1}&\cdots&\mathbf{B_i}&\cdots&\mathbf{B_n}&\mathbf{B_0}\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_i\\ \vdots\\ \mathbf{u}_n\\ \mathbf{u}_0\end{bmatrix} + \begin{bmatrix}d_1(k)\\ \vdots\\ d_i(k)\\ \vdots\\ d_n(k)\end{bmatrix} \tag{4.10}$$

and (4.10) can be further simplified to equation (4.11) as:

$$\mathbf{x}(k+1) = \mathbf{I}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{d}(k), \tag{4.11}$$

where $\mathbf{I} \in \mathbb{N}^{n\times n}$, $\mathbf{x}(k) \in \mathbb{N}^{n\times n}$, $\mathbf{B} \in \mathbb{N}^{n\times p}$, $\mathbf{u}(k) \in \mathbb{N}^{p}$, $\mathbf{d}(k) \in \mathbb{N}^{n}$, and $p = n(n+1)$.

Some remarks on the value of p are discussed in the sequel. The previous value of p is only a theoretical amount, since it assumes all mathematical possible connections among the compartments. If we do not permit the turning from a compartment to the same, the value of p has to be decreased with n. Thus, we obtain $p = n \cdot n$ for the maximal number of connections.

This problem can be considered in another way. If we have n pieces of points, we can connect them by $n \cdot n - n$ lines. The lines represent the turnings in the traffic, and points represent the compartments. In theoretical case each compartment has connection to the environment, therefore we have to increase the previous amount with n. We obtain $n \cdot n$ for the number of connections in the entire network, which coincides with the result of the another approach.

Now, the properties of matrix $\mathbf{B}$ are discussed.

1. All column sums are zero in $\mathbf{B}$, since always the same state appears in a column with diverse signs. This property reflects the mass-conservation, i.e. all vehicles that entered the network, will exit.
2. The row sums in each row are as follows:

$$\left(\sum_{\substack{l=1\\ l\neq i}}^{n} x_l(k)\right) - x_i(k) = \left(\sum_{l=1}^{n} x_l(k)\right) - 2x_i(k), \tag{4.12}$$

   where l denotes the number of compartments, and i denotes the row index in the matrix $\mathbf{B}$. We can define this as an asymmetric mass difference, or mass-balance indicator.
3. The i-th column is a 0 column in the part matrix $\mathbf{B}_i, i \neq 0$.
4. The part matrices $\mathbf{B}_i, i \neq 0$ are said to be *column conservation* matrices. A column conservation matrix can be obtained as a negative transpose of a *Laplacian* matrix [22]. Column conservation matrices have the following properties:
   a) They are negative semidefinite matrices (Definition 2.4).
   b) The 0 always appears as an eigenvalue.
   c) The vector $\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ is always an eigenvector of the column conservation matrices.
5. Taking points 3 and 4 into account we can state that all part matrices $\mathbf{B}_i, i \neq 0$ has at least two 0 eigenvalues. One is associated with the zero column, the other is obtained from the linearly dependent columns.
6. The number of nonzero columns in each part matrix $\mathbf{B}_i, i \neq 0$ is equal to the number of inflows (except the input demand) into the i-th compartment.
7. Consider the part matrix $\mathbf{B}_i, i \neq 0$ belongs to the i-th compartment. In this matrix the nonzero columns have the same index as the compartments which send mass to the i-th compartment.
8. The trace of the part matrix $\mathbf{B}_i, i \neq 0$ can be calculated as:

$$\mathrm{tr}(\mathbf{B}_i) = -\sum_{\substack{l=1\\ l\neq i}}^{n} x_l = -\sum_{l=1}^{n} x_l - x_i, \tag{4.13}$$

   where l denotes a counter for compartments.

4.3.2 Matrix equation form 2

We can factorize the states out to a detached vector from (4.8) in the presented way.

The other matrix form of model equation (4.4) can be written as follows:

$$\mathbf{x}(k+1) = \mathcal{A}\mathbf{x}(k) + \mathbf{d}(k) \tag{4.14}$$

where vectors are defined in the space $\mathbb{R}^n$, and $\mathcal{A} \in \mathbb{R}^{n \times n}$.

The terms of the equation 4.14 are the following matrices and vectors.

$$\mathcal{A} = \begin{bmatrix} 1 - \sum\limits_{\substack{j=1\\j\neq 1}}^{n} \beta_{j1}(k) - \beta_{01}(k) & \cdots & \beta_{1i}(k) & \cdots & \beta_{1n}(k) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \beta_{i1}(k) & \cdots & 1 - \sum\limits_{\substack{j=1\\j\neq i}}^{n} \beta_{ji}(k) - \beta_{0i}(k) & \cdots & \beta_{in}(k) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n1}(k) & \cdots & \beta_{ni}(k) & \cdots & 1 - \sum\limits_{\substack{j=1\\j\neq n}}^{n} \beta_{jn}(k) - \beta_{0n}(k) \end{bmatrix},$$

$$\mathbf{x}(k+1) = \begin{bmatrix} x_1(k+1) \\ \vdots \\ x_i(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix}, \quad \mathbf{x}(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_i(k) \\ \vdots \\ x_n(k) \end{bmatrix}, \quad \mathbf{d}(k) = \begin{bmatrix} d_1(k) \\ \vdots \\ d_i(k) \\ \vdots \\ d_n(k) \end{bmatrix}$$

The matrix $\mathcal{A}$ has the following properties.

1. The matrix $\mathcal{A}$ is a *compartmental* matrix since it fulfils the requirements of Definition 2.5.
2. The sums appear in the diagonal entries correspond the constraint in equation (4.2).
3. The column sums are $1 - \beta_{0i}$, $\forall i$.
4. The eigenvalues of matrix $\mathcal{A}$, denoted by λ, are its diagonal entries, representing the remaining rate of the vehicles in each compartment. The remaining rate of vehicles is definied by the formula: $1 - \sum_{j=1 j\neq i}^{n} \beta_{ji}(k)$. Thus, each eigenvalue of the matrix $\mathcal{A}$ satisfy the inequality $0 \leq \lambda_i \leq 1$, $\forall i$.
5. The trace of $\mathcal{A}$ is as follows,

$$\mathrm{tr}\mathcal{A} = \sum_{i=1}^{n} \left[1 - \sum_{\substack{j=1\\j\neq i}}^{n} \beta_{ji}(k) - \beta_{0i}(k) \right]. \tag{4.15}$$

   This amount represents the total number of remaining vehicles in the network at each time step ('before' the inflowing vehicles arrive).

6. The matrix $\mathcal{A}$ has full rank, if the condition $\sum_{\substack{j=0\\j\neq i}}^{n} \beta_{ji}(k) < 1$, $\forall i$ is fulfilled.

Based on the previous expositions we can define the discrete time compartmental traffic system with infinite capacity.

Definition 4.2 (Discrete time compartmental traffic system with infinite capacity) *The discrete time compartmental traffic system with infinite capacity is definied by the difference equation (4.11). At the same time for parameters β (Definition 4.1) constraints (4.2), and (4.3) have to be satisfied.*

Why can the defined model be a compartmental model? The mass-conservation is ensured by the definition of the parameters of leaving flow rates (Definition 4.1) and by the constraints (4.2), and (4.3). Due to these restrictions one cannot let flow out more vehicles from the compartments, than the present number in it. However, the compartments do not have infinite capacity. The model equation cannot treat this property, so we have to introduce other constraints to take this fact into account.

4.4 Capacity constraints

In the foregoing subsections we have assumed that the compartments have infinite capacity respecting the number of vehicles present in the compartment. Unlike this assumption, the real road links have finite capacity, thus capacity constraints are introduced for the compartments.

The constraints on β were discussed above. Other restrictions are required to introduce the finite capacity of the compartments. The introduced CTM (see Section 3.2) differs from the compartmental approach, but the restricting principles can also be applied here, therefore part of those are adopted in the sequel.

In the compartmental approach the first constraint in (3.11) is satisfied by conditions (4.2) and (4.3). To fulfil the constraints of (3.11) related to the transmission and the receiver compartment, new conditions have to be declared.

The finite capacity of the trasmission is ensured by the throughput of the transmission rate, calculated from other properties of the network.

Definition 4.3 (Transmission throughput rate)
Suppose only unit vehicles with length l_{veh}, which hold the same following distance (l_{fol}). Based on the above variables the transmission throughput rate from compartment j to compartment i can be obtained as,

$$r_{i,j}(k) = \frac{1}{\dfrac{l_{veh} + l_{fol}}{v_{max}}}, \tag{4.16}$$

where v_{max} denotes the permitted maximal velocity in unit $\left(\frac{m}{s}\right)$, and $[r_{ij}(k)] = \frac{veh}{s}$.

This throughput highly depends on time and the actual traffic situations, since in congested compartments the vehicles are not able to move with free flow speed (v_{max}). The defined transmission throughput rate can be used only in uncongested cases. Now, this value will be assumed constant for all compartments. Therefore, all connections have the same maximal throughput. This amount is only an upper bound on the real performance of the transmission, since its definition contains theoretical values. Theoretical, since the the traffic flow does not consist of uniform vehicles with uniform drivers with the same following distance and vehicle length. For instance, trucks and buses have significant different properties compared to the cars. If the control of the intersection would depend on this ultimate maximal transmission rate, we could not reach the expected flow capacity. To eliminate this inaccuracy, an error term will be introduced into the model (see Section 6.2).

It is important to note that this throughput rate is not the same as the maximal flow rate on a link (compartment), while the dimension of the two amounts is the same $\left(\frac{\text{veh}}{\text{time unit}}\right)$. The transmission rate describes the connection, the link capacity rate is related to the link (compartment) itself.

In the compartmental traffic model each flow was given in the unit of vehicles (veh). In favour of the consistency of units the transmission throughput rate, given in measure $\left(\frac{veh}{s}\right)$, has to be reduced to this measure. Hence we introduce the variable T_s which denotes the sample time, measured in seconds.

Definition 4.4 (Maximal transmission)
The maximal transmission from the compartment j to compartment i is given by the following formula:

$$r'_{i,j} = r_{i,j} T_s. \tag{4.17}$$

The maximal capacity of a compartment is assumed to be time invariant, and is in the sequel denoted by x_i^{max}. By the following condition can be considered the empty place in the receiver compartment. The next inequality has to be held to the input flow to i-th compartment at time step $k+1$,

$$q_i(k+1) \leq x_i^{max} - x_i(k). \tag{4.18}$$

Both the maximal transmission and the maximal capacity of a compartment limit the number of inflow vehicles. A minimum condition can be given between these values, similarly to (3.11) by,

$$q_i(k) \leq \min\{r'_{i,j}, x_i^{max} - x_i(k)\}. \tag{4.19}$$

The equation (4.19) can be written more detailed as,

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \beta_{ij}(k) x_j(k) \leq \min\{r'_{i,j}, x_i^{max} - x_i(k)\}. \tag{4.20}$$

Actually, the left-hand side of the expression (4.20) means the inflow of a compartment itself, the right-hand side contains restrictions to the inflow. There can arise such cases, if the inflow remains below the capacity constraints. To handle this fact, the peak number of the entering vehicles into a compartment ($\bar{q}_i(k)$) at time step k can be computed by the definition:

Definition 4.5 (Peak number of entering vehicles into a compartment)
The peak number of entering vehicles into compartment i at time step k is given by the condition:

$$\bar{q}_i(k) = \min\left\{ \sum_{\substack{j=1 \\ j\neq i}}^{n} \beta_{ij}(k) x_j(k), r'_{i,j}, x_i^{max} - x_i(k) \right\}. \tag{4.21}$$

4.5 A special connection

In compartmental traffic systems a special case of the connections can be found. A further consideration has to be introduced at the computing of the input and output flows at these connections. They are special, since at the entry point of these connections more than one connections start to other compartments, and at the same time, at the endpoint of these connections more than one connections join to the same compartment (shortly called as n:n connections). To demonstrate this special case, let us examine the depicted part of a compartmental system in Figure 4. The connection β_{bc} is considered as a special connection.

The compartment b obtains input from the compartments a and c. The compartment d has one input, arriving from the compartment c. The capacity of each transmission is supposed to be the same, i.e. $r'_{i,i+1} = r$, $\forall i$. How should the input and output flows be computed, if the second or the third term in condition (4.21) is in action in point of compartment b?

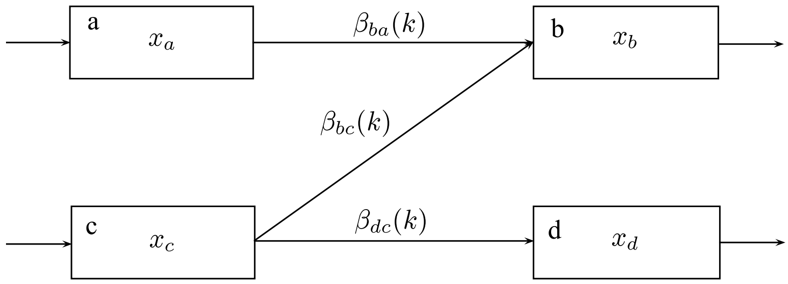

Figure 4: Network for demonstrating the special connection

At time step $k+1$ we assume that $r < \sum_{\substack{j=1 \\ j\neq b}}^{n} \beta_{bj}(k)x_j(k) < x_b^{max} - x_b(k)$. Thus the inflow of the compartment b compartment will become r. This amount has to be divided into two parts, an input from compartment a, and one from compartment c. More exactly, from each neighbouring compartment an amount $h_{bj}(k) = \frac{r}{\sum_j \beta_{bj}(k)x_j(k)}\beta_{bj}(k)x_j(k)$ arrives to compartment b, where $j \in \{a, c\}$. If we use this method, the proportions of β parameters are still considered, and no more vehicles can flow out from a compartment than the number present in it. Naturally, if the term $x_b^{max} - x_b(k)$ is the smallest one, the output of the compartments a and c can be obtained as $h_{bj}(k) = \frac{x_b^{max} - x_b(k)}{\sum_j \beta_{bj}(k)x_j(k)}\beta_{bj}(k)x_j(k)$.

Remark: this special connection is not permitted in the CTM approach. The connection is divided into two connections by the help of an artifical cell [8]. We cannot relate any real network part to this artifical cell, it is only required for the modeling.

Now we are in the position to define the discrete time compartmental traffic system with finite capacity.

Definition 4.6 (Discrete time compartmental traffc system with finite capacity)
The discrete time compartmental traffic system with finite capacity is defined by the difference equation (4.9). At the same time for parameter β (Definition 4.1) constraints in (4.2), and (4.3) have to be satisfied. Furthermore, condition (4.21) has to be held.

4.6 Simulation examples

In this section, we analyse a small-scale traffic network from compartments point of view. The network topology is depicted in Figure 6. We suppose that only the depicted mass transmissions have nonzero leaving flow rate. For the sake of simplicity the parameters β are supposed to be time-invariant and can be seen in Table (1). The disturbances ($d_1(k)$, $d_5(k)$, $d_8(k)$) are the same step functions, depicted in Figure 5. The model equation of this example network is presented in the sequel.

β_{21}	β_{32}	β_{45}	β_{62}	β_{64}	β_{74}	β_{78}	β_{03}	β_{06}	β_{07}
0.8	0.5	0.9	0.5	0.3	0.6	0.9	1	1	1

Table 1: Leaving flow rates

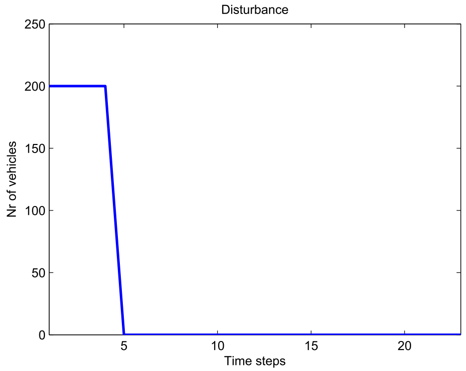

Figure 5: Disturbances

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \\ x_6(k+1) \\ x_7(k+1) \\ x_8(k+1) \end{bmatrix} = \begin{bmatrix} 1-\beta_{21}(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{21}(k) & 1-\beta_{32}(k)-\beta_{62}(k) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_{32}(k) & 1-\beta_{03}(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\beta_{64}(k)-\beta_{74}(k) & \beta_{45}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\beta_{45}(k) & 0 & 0 & 0 \\ 0 & \beta_{62}(k) & 0 & \beta_{64}(k) & 0 & 1-\beta_{06}(k) & 0 & 0 \\ 0 & 0 & 0 & \beta_{74}(k) & 0 & 0 & 1-\beta_{07}(k) & \beta_{78}(k) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\beta_{78}(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \\ x_7(k) \\ x_8(k) \end{bmatrix} + \begin{bmatrix} d_1(k) \\ 0 \\ 0 \\ 0 \\ d_5(k) \\ 0 \\ 0 \\ d_8(k) \end{bmatrix} \tag{4.22}$$

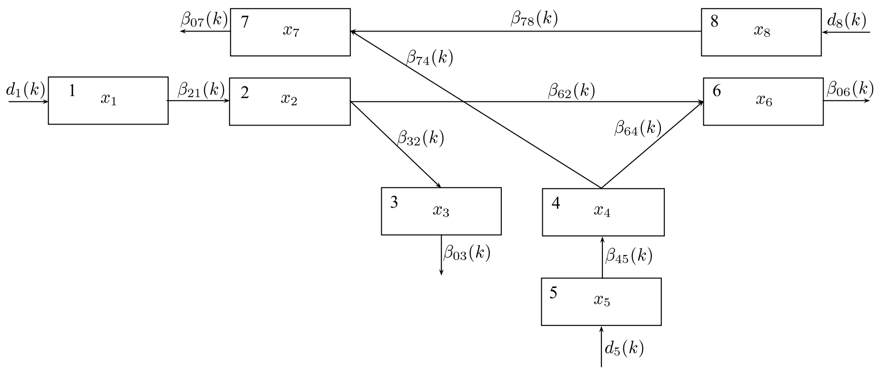

Figure 6: Example network

Two time domain simulations have been prepared over this network topology. In the first case, each compartment has infinite capacity, so the system matches Definition 4.2. The second case is based on Definition 4.6, thus the system was assumed as a finite capacity compartmental system. Capacity constraints were introduced as follows. Maximal transmission $r'_{i,i+1}$, $\forall i$ is time-invariant, and has a value of 100 vehicles per time step. The maximal compartment capacity ($x_i^{max} \quad \forall i$) is supposed to be time-invariant with rate equals to 100 vehicles, as well. The input compartments 1, 5, 8 have infinite capacity, similarly to the CTM approach. The simulation results with the infinite capacity case can be seen in Figure 7 and the results with finite capacity in Figure 8, respectively.

The infinite capacity system empties three times faster than the finite one. In the latter case all states have oscillation, since the empty compartment is filled with vehicles in one step. Definitely, in the next step it will be emptied. As it can be seen in Figure 8, two types of oscillation can be separated. The state variable x_7 changes its value between the possible borders, namely between zero and the maximal capacity (x_7^{max}). The other type of oscillation is presented by the state variable x_4, which remains between two intermediate values.

The explanation is very simple. If at the second time step the number of entering vehicles is equal to (or would be more than) the maximal capacity of the considered compartment, the appropriate state variable reaches the upper border. At the third time step zero number of vehicles can enter the compartment, since it is saturated. At the same time, all of the present vehicles can leave it. At time step four the maximal number of vehicles can flow in again. This process continues until more vehicles would like to enter the compartment, than it can receive.

Now let us consider the other oscillation. The number of entering vehicles into compartment i is denoted by $q_i(2)$ at time step 2. The inequality $q_i(2) < x_i^{max}$ is supposed. Thus, there is empty space in the compartment i, which has a value of $x_i^{max} - q_i(2)$. At the third time step the vehicle set $q_i(2)$ leaves the compartment, and $x_i^{max} - q_i(2)$ can enter it. At the fourth time step the latter value will be the number of leaving vehicles. This oscillation goes on until more vehicles would like to enter the compartment, than it can receive.

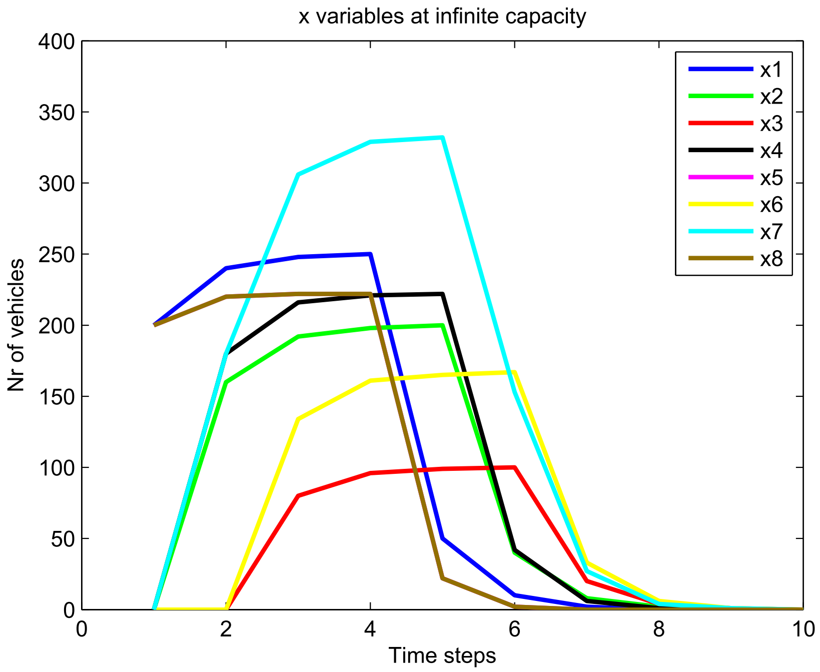

Figure 7: Results with infinite capacity

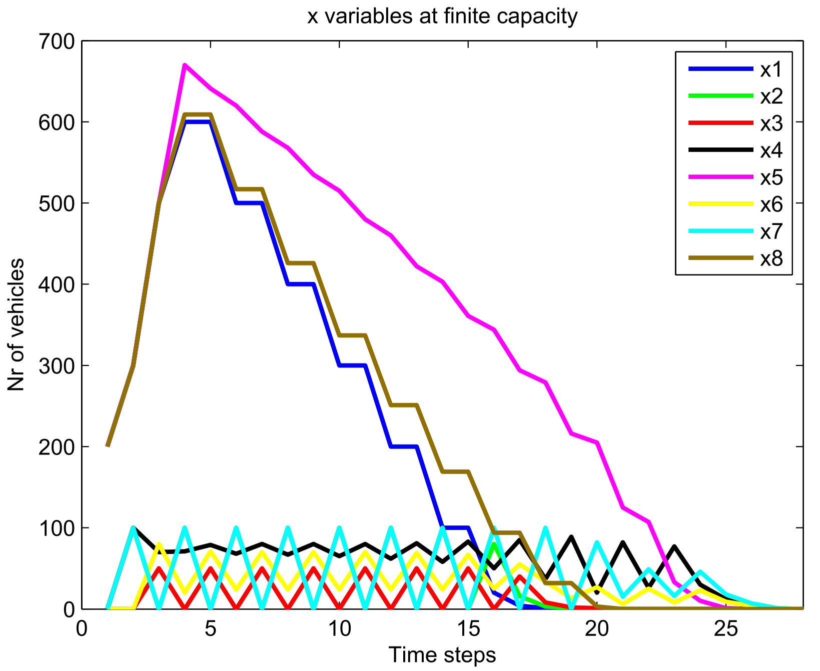

Figure 8: Results with finite capacity

Let us examine how the state variables change, if the outflow to the environment is bounded, i.e. for the parameters $\beta_{0i}(k)$ the equality $\beta_{0i}(k) = 1$, $\forall i$ not valid. β_{03}, β_{06}, and β_{07} are modified, as it can be seen in Table 2.

β_{21}	β_{32}	β_{45}	β_{62}	β_{64}	β_{74}	β_{78}	β_{03}	β_{06}	β_{07}
0.8	0.5	0.9	0.5	0.3	0.6	0.9	**0.8**	**0.5**	**0.6**

Table 2: Modified leaving flow rates

The matrix $\mathcal{A}$ can be written by using the above β-s as,

$$\mathcal{A} = \begin{bmatrix} 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.3 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 & 0.9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 \end{bmatrix}.$$

The results are depicted in Figure 9.

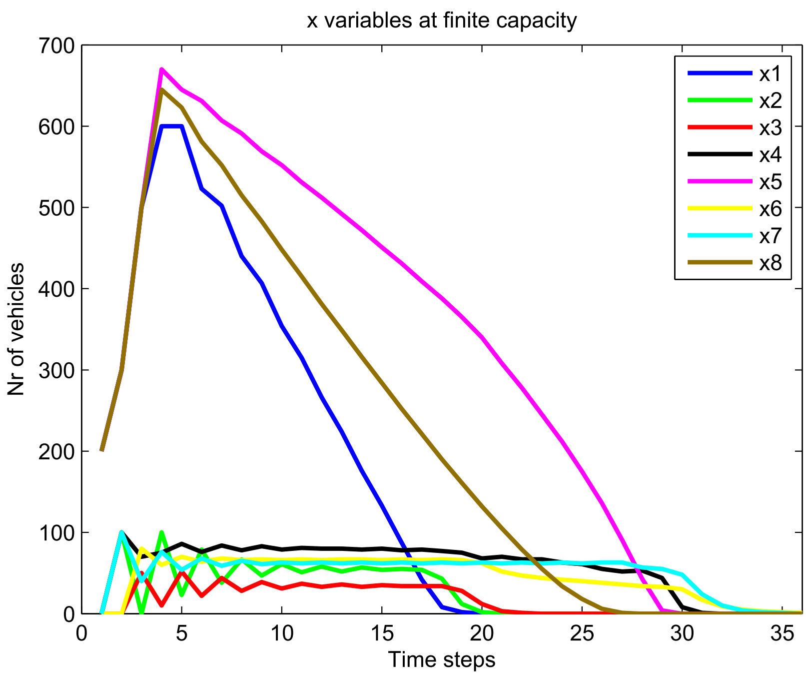

Figure 9: Results with finite capacity and modified leaving flow rates

As it was expected, there are no more long oscillations at any state variable. The exhaustion needs more time compared to the case with unbounded environment outflow. As the parameters β_{0i} approach even the value of 1, even more oscillation will be produced by the system. At low leaving flow rates to the environment the state variables reach a constant value after some transient.

This value is held until the number of vehicles - intending to inflow these compartments - is greater than the maximal capacity of the compartments.

The exhaustion is demonstrated by the following small example. Let us assume that $\beta_{01} = 0.5$, and the peak number of entering vehicles to compartment 1 is $\overline{q}_1 = 100$ vehicles per time step, time independently. Initially compartment 1 is empty, i.e. $x_1(0) = 0$. At the first time step the peak number of entering vehicles can flow into compartment 1: $x_1(1) = 100$. At the second time step the compartment is full, thus no vehicles can flow into it, but half of the present vehicles flow out: $x_1(2) = 50$. At the third step 50 vehicles flow in, and the half of the present 50 vehicles flow out: the result is $x_1(3) = 75$. The fourth time step changes the state x_1 as: 25 vehicles flow in, and 37 vehicles flow out, the remaining amount is $x_1(4) = 63$ vehicles. This method can be carried on, the 'equilibrium' number will be 65-66.

By the following examples can we examine how is affected the outflow by choosing one outflow to the environment as 0. The compartments without any output are said traps, [24]. The referred thesis contains conditions and theorems about the existence of traps. Firstly, $\beta_{06} = 0$ has been applied, in the second case $\beta_{07} = 0$ was chosen. The other leaving flow rates have a value between 0 and 1, as it is given in Table 3.

β_{21}	β_{32}	β_{45}	β_{62}	β_{64}	β_{74}	β_{78}	β_{03}	β_{06}	β_{07}
0.8	0.5	0.9	0.5	0.3	0.6	0.9	0.8	**0**	0.6
0.8	0.5	0.9	0.5	0.3	0.6	0.9	0.8	0.5	**0**

Table 3: Modified leaving flow rates

Two inflows can be recognized at compartment 6. It receives vehicles from compartments 2 and 4. If we prohibit the outflow from compartment 6, i.e. a trap is constructed, the behaviour of the network changes. As it was expected, compartment 6 reaches its capacity, and this state does not decrease during the entire simulation. The trap compartment keeps the vehicles in itself. By comparing Figure 9 to Figure 10 can be stated that the time required to empty the non-trap compartments is greater by a multiplier 1.5, than in the trapless case. The compartments, which send vehicles to compartment 6, are connected to other compartments, as well. Therefore, after the saturation of compartment 6 they do not become saturated. The outflow from compartment 4 to compartment 7, and the outflow from compartment 2 to compartment 3 still trasmit vehicles, thus the compartments 4 and 2 will be empty, but after a longer time.

Let us consider the Figure 11, which depicts the results obtained by prohibit the outflow from compartment 7. In this case compartment 8 cannot send any vehicle to other compartments, since it has only one outflow to compartment 7, which cannot receive more mass after its saturation. The degree of the diminish is smaller in compartments 4 and 5, since they are connected to the trap compartment.

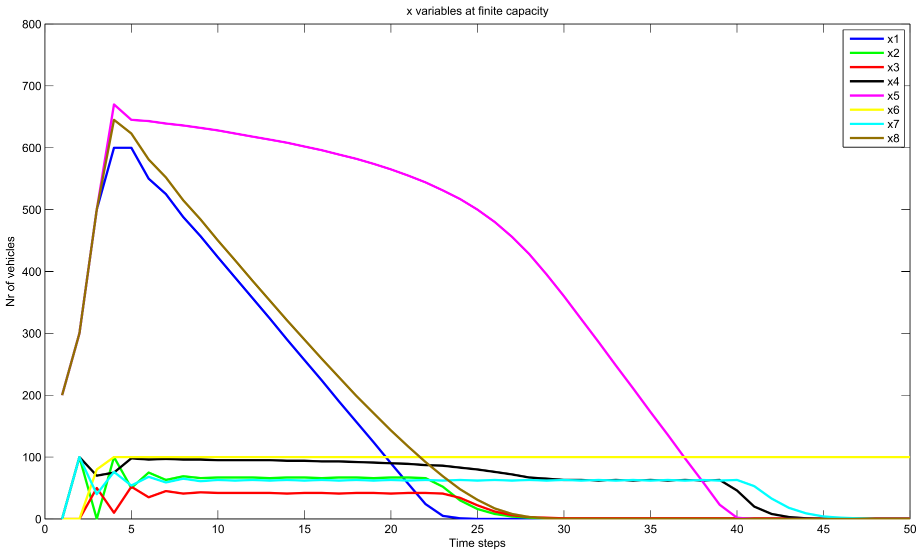

Figure 10: Results with trap at compartment 6

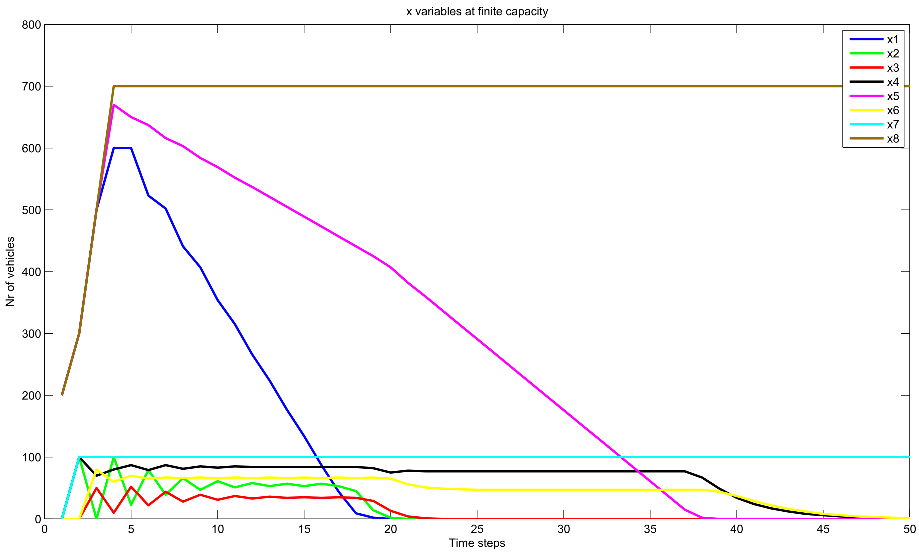

Figure 11: Results with trap at compartment 7

5 Model analysis

The presented compartmental traffic model is examined in point of view the stability, controllability, diagonalizability, and eigenvalue sensitivity in this section. The knowledge of these parameters is required in control synthesis.

5.1 Stability

Initially, we will distinguish between two cases in stability analysis. Firstly, the compartmental traffic system is examined with constant parameters β. Secondly the parameters β are allowed to change. Actually, the latter case corresponds the real control method.

5.1.1 Stability analysis with constant β

Let us examine the stability property of the autonomous compartmental traffic system in Lyapunov-sense (Definitions 2.14 and 2.15). Therefore, the system equation (4.14) is modified. It is assumed - for the reason of stability analysis - that all disturbance inputs are zero, thus the difference equation is simplified to:

$$\mathbf{x}(k+1) = \mathcal{A}\mathbf{x}(k). \tag{5.1}$$

In stability analysis it is common to introduce a quadratic Lyapunov function to adress stability by dissipativity [22]. (The Lyapunov function is usually denoted by V.) The general quadratic form is $V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbf{P}\mathbf{x}(k)$, where $\mathbf{P}$ is a positive-definite matrix (Definition 2.3). Instead of this form, at compartmental systems we can choose a linear Lyapunov function, as it is stated in [5]. The Lyapunov-function is always constructed by such a parameter, which can express the changings in the system, and it is expected that its amount decreases, if the system approaches the equilibrium point. The amount of mass in a compartmental system meets these requirements.

$$V(\mathbf{x}(k)) = \mathbf{e}^T\mathbf{x}(k), \tag{5.2}$$

where $\mathbf{e}^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^n$. Practically, in this special case the Lyapunov function represents the sum of the mass (number of vehicles) in the network at time step k. Consider the difference $\Delta V(\mathbf{x}(k))$ as,

$$\Delta V(\mathbf{x}(k)) = V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)), \tag{5.3}$$

$$\Delta V(\mathbf{x}(k)) = \mathbf{e}^T\mathcal{A}\mathbf{x}(k) - \mathbf{e}^T\mathbf{I}\mathbf{x}(k), \tag{5.4}$$

$$\Delta V(\mathbf{x}(k)) = \mathbf{e}^T(\mathcal{A} - \mathbf{I})\mathbf{x}(k), \tag{5.5}$$

where $\mathbf{I}$ is the n-dimensional identity matrix. The compartmental system given by (5.1) is stable, if $\Delta V < 0$, $\forall \mathbf{x}(k) \neq 0$. After some calculation we can obtain:

$$\Delta V(\mathbf{x}(k)) = \begin{bmatrix} -\beta_{01}(k) & \cdots & -\beta_{0i}(k) & \cdots & -\beta_{0n}(k) \end{bmatrix} \mathbf{x}(k). \tag{5.6}$$

In this consideration, $\Delta V(\mathbf{x}(k))$ always equals to the outflow to the environment. The amount $\Delta V(\mathbf{x}(k))$ is always negative, since (4.2) is valid for the entries of the above row vector, if $\beta_{0i}(k) \neq 0$ for at least one i. More, the entries of $\mathbf{x}(k)$ are always nonnegative, since the compartmental traffic system is a nonnegative system (Definition 2.11). It follows that the compartmental system without disturbances is stable. If the entire mass can flow out through an exit, the equilibrium point is zero. Zero is a stable equilibrium point for such a system ($\mathbf{x}_e = 0$). If the autonomous system is disturbed with finite energy, the meaning of the stability is to returned to the equilibrium point.

On the other hand, if there are compartments in the network without outflow (i.e. traps), the equilibrium point will be the sum of the mass in these compartments.

The traps can play an important role in compartmental traffic systems. They can be considered as parking lots, if - exceptionally in this case - reverse traffic flow is permitted from the trap compartments to the neighbouring compartments, which connections represent the re-entering of the vehicles to the road network.

The observations can be formalized as follows. Let us denote by T the set of compartments, which does not have an output to the environment or to other compartments, and by $\mathbf{x}_e$ the equilibrium point of the system. There is such a notation of compartments, in which compartments without output have subscripts $i, i+1, \cdots, i+m$, if there exist any compartment with such a property. If the following equality is held in $k \to \infty$, $\mathrm{T} = \{\emptyset\}$, for the equilibrium point $\mathbf{x}_e = 0$ is fulfilled.

Proposition 5.1
If $\mathrm{T} = \{i, i+1, \cdots, i+m\}$, *the equilibrium point of the autonomous system can be obtained as* $\mathbf{x}_e = \sum_{j=i}^{i+m} x_j$, *where* $k \to \infty$. *This equilibrium can be considered as unstable equilibrium with infinite capacity compartments.*

The latter case can be demonstrated by the following small example (Figure 12).

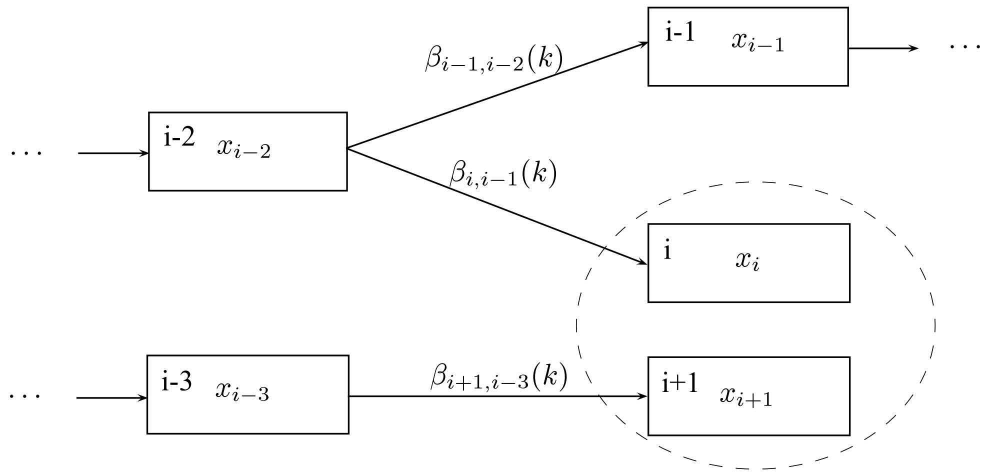

Figure 12: Compartments without output

The mass can flow from compartment $i-2$ to compartments $i-1$ and i. Similarly, compartment $i-3$ give mass to compartment $i+1$. Reverse directional flow is not permitted in this network. The compartment $i-1$ has a connection to the environment, thus the mass can flow out from it. But compartment i and $i+1$ has only an input connection. If we stop the input to this network, after a certain time compartments $i-3$, $i-2$, and $i-1$ will empty, but the compartment i and $i+1$ will not. (It is assumed, that compartment $i+1$ has greater, or infinite capacity than compartment $i-3$.)

According to [5] the compartmental system is asymptotically stable, if $|\mathcal{A}| \neq 0$, i.e. the matrix $\mathcal{A} \in \mathbb{R}^n$ has n eigenvalues different from zero (Property 6 on page 18). In other words, if by means of network traps the vehicle mass is cummulating the network will not be emptied, and $\mathbf{x} = 0$ is not a feasible and stable equilibrium, as it was before.

Let us consider another approach to obtain information about the behaviour of the compartmental system. If the initial state vector $\mathbf{x}(0)$ can be written as a linear combination of the eigenvectors of the system matrix $\mathcal{A}$ as,

$$\mathbf{x}(0) = \sum_{j=1}^{n} c_i \mathbf{v}_i, \tag{5.7}$$

where c_i denotes constant values, and $\mathbf{v}_i$ denotes the eigenvectors of the matrix $\mathcal{A}$. The dynamical equation of the autonomous system can be obtained from (4.14), for $\mathbf{x}(1)$ we come to the following equation by substituting the previous formula:

$$\mathbf{x}(1) = \mathcal{A} \sum_{j=1}^{n} c_i \mathbf{v}_i. \tag{5.8}$$

It is required to make the previous step that the matrix $\mathcal{A}$ has n different eigenvectors (i.e. n eigenvalues different from zero), which means in this case that any compartment cannot have $\sum_{j=0j\neq i}^{n} \beta_{ji} = 1, \forall i$.

Expanding the sum in (5.8) can be obtained as,

$$\mathbf{x}(1) = \mathcal{A} c_1 \mathbf{v}_1 + \cdots + \mathcal{A} c_n \mathbf{v}_n. \tag{5.9}$$

Considering the Definition 2.2, we further simplify (5.9),

$$\mathbf{x}(1) = \lambda_1 \mathbf{I} c_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{I} c_n \mathbf{v}_n. \tag{5.10}$$

The right-hand side of equation (5.10) can be formalized as the product of two vectors as,

$$\mathbf{x}(1) = \begin{bmatrix} \lambda_1 \mathbf{I} & \cdots \lambda_n \mathbf{I} \end{bmatrix} \begin{bmatrix} c_1 \mathbf{v}_1 \\ \vdots \\ c_n \mathbf{v}_n \end{bmatrix}. \tag{5.11}$$

(The unit matrix provides the appropriate matrix dimensions.) By progressing the time to the time step k we obtain the state vector as,

$$\mathbf{x}(k) = \mathcal{A}^k \mathbf{x}(0). \tag{5.12}$$

The equation (5.12) can be cast into the form of (5.11) as,

$$\mathbf{x}(k) = \begin{bmatrix} \lambda_1^k \mathbf{I}^k & \cdots \lambda_n^k \mathbf{I}^k \end{bmatrix} \begin{bmatrix} c_1 \mathbf{v}_1 \\ \vdots \\ c_n \mathbf{v}_n \end{bmatrix} \tag{5.13}$$

We know from the eigenvalues of the matrix $\mathcal{A}$ that they fall to the range [0 1] (Property 4 on the page 18). Therefore, the vector of λ-s will have even smaller entries by the continuous multiplication. Thus, the entries of the state vector of the autonomous system will decrease continuously in the future, i.e. the number of vehicles will diminsh during the time resulting in a stable system.

If there is a trap in the compartmental network (i.e. one or more compartments do not have any output), the eigenvalue belonging to the trap has the value of 1. This entry will not decrease by the multiplication of the vector of λ-s by itself. Thus, the corresponding value $c_i \mathbf{v}_i$ will hold its initial value, i.e. the number of vehicles does not decrease.

5.1.2 Stability analysis with changing β

The stability derived from the remaining number of vehicles can be computed only pointwisely, if we do not assume any pattern in β-changes. Because of the change of the parameters β, the equilibrium point changes its value at each time step. Asymptotically stable condition cannot be stated in this case.

If the compartmental network does not contain any traps, the local equilibrium can, however, be ensured. Urban traffic networks (without parking lots) can be considered as trap-free networks, but traps may be produced by the parameters β-s. Theoretically, β_{ij} can have its value between 0 and 1. The j-th compartment with $\beta_{ij} = 0$ can cause the arise of traps. But if we introduce minimal green times greater than zero as constraints, it can be easily seen that to this minimal green time cannot correspond the $\beta_{ij} = 0$, but a $\beta_{ij} > 0$. Thus, a trap-free network cannot be converted into a trap-network.

5.2 Controllability

The reachability of discrete time nonlinear systems is given by Definition 2.16. This definition can be simplified to Definition 2.17 at LTI systems. Remark 2.18 builds a connection between the reachability and controllability properties. At discrete time bilinear systems matrices can be composed to examine the controllability of the system, which are diverse from the controllability matrices of linear systems.

Proposition 5.2
The matrix $\mathcal{A}$ can be inverted, if it has n linearly independent eigenvectors (n eigenvalues different from zero). In case of the compartmental traffic systems the matrix $\mathcal{A}$ has n linearly independent eigenvectors, if any compartments do not have the property $\sum_{j=0j\neq i}^{n} \beta_{ji} = 1$.

In order to give a necessary and sufficient condition for local controllability, a decomposition of the dynamical equation (4.11) is required [11]. More specifically, we have to rewrite the bilinear term into a sum, as it can be seen in equation (5.14):

$$x(k+1) = \mathbf{A}\mathbf{x}(k) + \sum_{l=1}^{p} u_l(k)\mathbf{Q}_l\mathbf{x}(k) + \mathbf{d}(k), \tag{5.14}$$

where $p = n \cdot n$ in case of the compartmental traffic systems.

Our system can be cast into the above form, started from the second matrix equation representation (4.14). The terms 1 in the diagonal can be stacked into a distinct matrix. Thus, the remaining matrix contains only $\beta_{ij}(k)$ values, and the unit matrix correponds the matrix $\mathbf{A}$ in (5.14). With this reformulation we obtain,

$$
\begin{bmatrix} x_1(k+1) \\ \vdots \\ x_i(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_i(k) \\ \vdots \\ x_n(k) \end{bmatrix} +
$$

$$
+ \begin{bmatrix} -\sum\limits_{\substack{j=1 \\ j\neq 1}}^{n} \beta_{j1}(k) - \beta_{01}(k) & \cdots & \beta_{1i}(k) & \cdots & \beta_{1n}(k) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \beta_{i1}(k) & \cdots & -\sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \beta_{ji}(k) - \beta_{0i}(k) & \cdots & \beta_{in}(k) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n1}(k) & \cdots & \beta_{ni}(k) & \cdots & -\sum\limits_{\substack{j=1 \\ j\neq n}}^{n} \beta_{jn}(k) - \beta_{0n}(k) \end{bmatrix} \cdot \qquad (5.15)
$$

$$
\cdot \begin{bmatrix} x_1(k) \\ \vdots \\ x_i(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} d_1(k) \\ \vdots \\ d_i(k) \\ \vdots \\ d_n(k) \end{bmatrix}
$$

By reason of size limits the term with the sum in (5.14) will not be detailed, but we show one term from this sum. The entry intersected by the first row and the i-th column can be distributed as follows $\forall l \in \{1 \cdots p\}$,

$$
\underbrace{\beta_{1l}(k)}_{u_l} \underbrace{\begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}}_{\mathbf{Q}_l} \underbrace{\begin{bmatrix} x_1(k) \\ \vdots \\ x_i(k) \\ \vdots \\ x_n(k) \end{bmatrix}}_{\mathbf{x}} . \qquad (5.16)
$$

The matrix $\mathbf{Q}_l$ is a square matrix with dimension n, and contains only ± 1 and 0. We can find 1 in that entry of the matrix $\mathbf{Q}_l$, where u_l has a positive value in matrix $\mathcal{A}$, i.e. the compartment denoted by the row index receives mass from the compartment denoted by the column index of matrix $\mathbf{Q}_l$. Similarly, the -1 value stands in such a position in the matrix $\mathbf{Q}_l$, where the control input u_l has a negative sign in the matrix $\mathcal{A}$, i.e. the compartment denoted by the row index sends mass to the compartment denoted by the column index of matrix $\mathbf{Q}_l$. In the diagonal entries of $\mathbf{Q}_l$ only -1 and 0 can stand, and in the off-diagonal field we can find only 1 and 0 values.

In case of $\beta_{ij}(k) \quad i \neq 0$ the matrix $\mathbf{Q}_l$ contains only one -1 in the diagonal, and a single 1 in the same column when the -1 stands. In case of $\beta_{ij}(k)$, $i = 0$ the matrix $\mathbf{Q}_n$ contains only one -1 in its diagonal.

Definition 5.3 (Local controllability of discrete-time bilinear systems [11])
The discrete-time bilinear system

$$x(k+1) = \mathbf{A}\mathbf{x}(k) + \sum_{l=1}^{p} u_l(k)\mathbf{Q}_l\mathbf{x}(k) \tag{5.17}$$

is locally controllable on the interval $[0, N]$ *if and only if* rank $\mathcal{C} = n$, *where*

$$\mathcal{C} = \Big[\mathbf{Q}_1\mathbf{A}^{N-1}\mathbf{x}(0) \quad \mathbf{A}\mathbf{Q}_1\mathbf{A}^{N-2}\mathbf{x}(0) \quad \cdots \tag{5.18}$$

$$\mathbf{A}^{N-1}\mathbf{Q}_1\mathbf{x}(0) \quad \cdots \quad \mathbf{Q}_p\mathbf{A}^{N-1}\mathbf{x}(0) \quad \mathbf{A}\mathbf{Q}_p\mathbf{A}^{N-2}\mathbf{x}(0) \quad \cdots \quad \mathbf{A}^{N-1}\mathbf{Q}_p\mathbf{x}(0)\Big], \tag{5.19}$$

where $\mathbf{x}(0) \neq \mathbf{0}$, *with a control sequence satisfying that,*

$$\left[\mathbf{A} + \mathbf{u}_1\mathbf{Q}_1 + \cdots + \mathbf{u_p}\mathbf{Q}_p + \mathbf{Q}_1\mathbf{x}(k)\frac{\partial \mathbf{u}_1}{\partial \mathbf{x}(k)} + \cdots + \mathbf{Q}_m\mathbf{x}(k)\frac{\partial \mathbf{u}_p}{\partial \mathbf{x}(k)}\right] \tag{5.20}$$

has full rank ($\mathcal{C}$ *has dimension* $n \times Np$). *The last matrix has to have full rank at each time step* k.

Let us build the matrix $\mathcal{C}$ from the matrices of the compartmental traffic system. Since the matrix $\mathbf{A}$ is a unit matrix in this approach, its powers are also unit matrices. The matrix $\mathcal{C}$ will be simpler, as it can be seen in the following formula:

$$\mathcal{C} = \Big[\underbrace{\mathbf{Q}_1\mathbf{x}(0) \quad \cdots \quad \mathbf{Q}_1\mathbf{x}(0)}_{\text{N}}\ \underbrace{\mathbf{Q}_2\mathbf{x}(0) \quad \cdots \quad \mathbf{Q}_2\mathbf{x}(0)}_{\text{N}} \cdots \underbrace{\mathbf{Q}_p\mathbf{x}(0) \quad \cdots \quad \mathbf{Q}_p\mathbf{x}(0)}_{\text{N}}\Big]. \tag{5.21}$$

In this hypermatrix N-N columns are the same. Recall that there is only one non-zero column in each $\mathbf{Q}_l$. In general case there are $n \cdot n$ different piece of matrix $\mathbf{Q}_l$, thus there are $n \cdot n$ non-zero columns in the matrix $\mathcal{C}$. These columns are linearly dependent. There are n pieces of columns (these belong to the connections to the environment), in which only one -1 stands in the diagonal. This set of columns is linearly independent, and can be considered as a basic vector system for $\mathcal{C}$. All other column can be expressed by a linear combination of the elements of the basis. For instance, let us consider a compartmental network with 3 compartments, and all possible connections, as it is depicted in Figure 13.

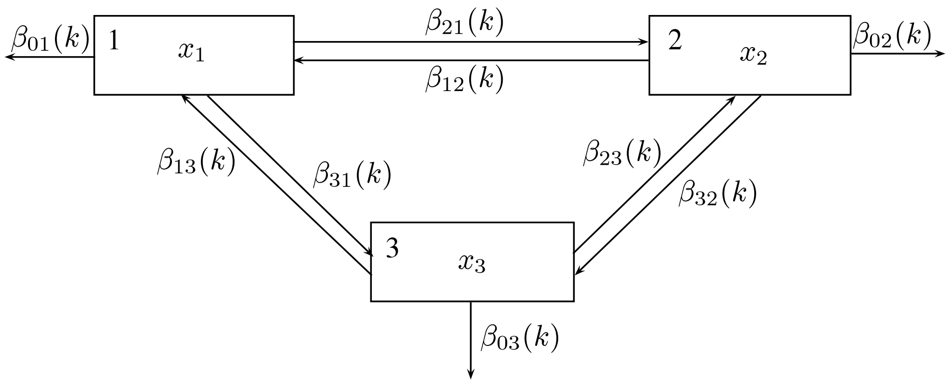

Figure 13: Compartmental network with all possible connections

The $\mathbf{Q}$ matrices are given by,

$$\begin{bmatrix}-1&0&0\\0&0&0\\1&0&0\end{bmatrix}\quad\begin{bmatrix}-1&0&0\\1&0&0\\0&0&0\end{bmatrix}\quad\begin{bmatrix}0&1&0\\0&-1&0\\0&0&0\end{bmatrix}\quad\begin{bmatrix}0&0&0\\0&-1&0\\0&1&0\end{bmatrix}\quad\begin{bmatrix}0&0&0\\0&0&1\\0&0&-1\end{bmatrix}\quad\begin{bmatrix}0&0&1\\0&0&0\\0&0&-1\end{bmatrix}$$
$$\begin{bmatrix}-1&0&0\\0&0&0\\0&0&0\end{bmatrix}\quad\begin{bmatrix}0&0&0\\0&-1&0\\0&0&0\end{bmatrix}\quad\begin{bmatrix}0&0&0\\0&0&0\\0&0&-1\end{bmatrix}.$$

The above matrices span a convex hull, which coincidences the hull of the controllability matrix $\mathcal{C}$. The following matrix contains the nonzero columns of the matrix $\mathcal{C}$.

$$\begin{bmatrix}-1&-1&1&0&0&1&-1&0&0\\1&0&-1&-1&1&0&0&-1&0\\0&1&0&1&-1&-1&0&0&-1\end{bmatrix}\mathbf{x}(0) \tag{5.22}$$

It seems clear that the first six columns in the matrix (5.22) can be made as a linear combination of the last three columns, which are linearly independent. Moreover, these vectors compose an orthonormal basis. Thus, it is valid for the matrix $\mathcal{C}$ that rank $\mathcal{C} = n = 3$, which corresponds to Definition 5.3.

If we would produce a trap in this compartmental network by setting the leaving flow rates $\beta_{ji}(k)$, $\forall j \neq 0$ of any compartment, the controllability of the network would not change. Two matrices $\mathbf{Q}$ would be zero matrix, but the last three matrices $\mathbf{Q}$ would not be affected. By choosing one or two $\beta_{0i} = 0$, the equality: rank $\mathcal{C} = 3$ is valid. It can be easily seen that to network topologies without any outflow to the environment can be related a controllability matrix without full rank. For instance, the last three columns in the matrix (5.22) are full-zero columns. Three pairs of columns can be found, where the sum of vectors is the zero vector. Thus, the columns first, second, and fourth remain. The fourt can be expressed as the difference of the other two. We can found only two linearly independent columns, which means that the controllability matrix $\mathcal{C}$ does not have full rank.

Let us consider the condition related to the control input sequence. In the matrix (5.20) the u_i terms are independent of x, since they are only the β_{ij} values. Therefore $\frac{\partial u_i(k)}{\partial \mathbf{x}} = 0$, $\forall i$. Thus, the last p term of this matrix is zero. The remaining part is

$$[\mathbf{A} + u_1\mathbf{Q}_1 + \cdots + u_p\mathbf{Q}_p] \tag{5.23}$$

which coincides the matrix $\mathcal{A}$ in our model. The full rank condition is given formerly among the properties of matrix $\mathcal{A}$ (Property 6, on page 18).

Basically, each compartmental traffic system with a 'normal' structure is controllable. However, by changing the control inputs, the system can become uncontrollable in a couple of ways. Namely:

1. Each outflow to the environment β_{0i} $\forall i$ becomes zero.
2. So many β-s become zero, that there are less connections with $\beta_{ij} \geq 0$, than the number of the compartments.
3. All of the β-s related to a compartment becomes zero at the same time. (The disturbance cannot be considered as inflow.)
4. Some connections become zero such a way that the entire network splits into two or more independent parts, which parts are not connected to each other.

The last scenario is depicted in the Figure 14. If the β_{32} changes to 0, the compartments 1-2 and the 3-7 are detached from each other.

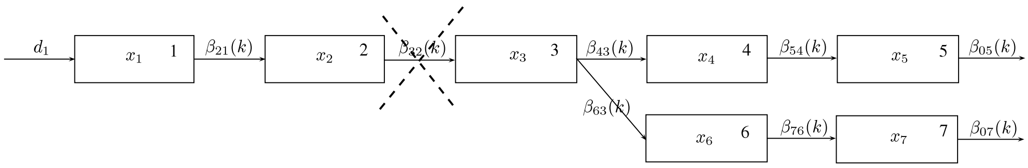

Figure 14: Uncontrollable network

5.3 Diagonalizability

With the help of this technique the original system can be decoupled into subsystems, which have their own inputs, disturbances, and control. Actually, the diagonalization is a state-space transformation in the dynamic description. In the diagonal representation the system matrix $\mathcal{A}$ has a diagonal form, which is denoted by $\overline{\mathcal{A}}$, in the sequel. To transform the original state-space to diagonal, a transformation matrix is required. The transformation is defined as follows

$$\mathbf{x} = \mathbf{T}\overline{\mathbf{x}} \tag{5.24}$$

where $\mathbf{T}$ is the transformation matrix, and $\overline{\mathbf{x}}$ is the state vector in diagonal representation. The state-space transformation exists, if the transformation matrix $\mathbf{T}$ has full rank, i.e. it is invertible. Recall the model difference equation (4.14):

$$\mathbf{x}(k+1) = \mathcal{A}\mathbf{x}(k) + \mathbf{d}(k). \tag{5.25}$$

Substituting (5.24) into the model equation yields:

$$\mathbf{T}\overline{\mathbf{x}}(k+1) = \mathcal{A}\mathbf{T}\overline{\mathbf{x}}(k) + \mathbf{d}(k), \tag{5.26}$$

$$\overline{\mathbf{x}}(k+1) = \mathbf{T}^{-1}\mathcal{A}\mathbf{T}\overline{\mathbf{x}}(k) + \mathbf{T}^{-1}\mathbf{d}(k), \tag{5.27}$$

$$\overline{\mathbf{x}}(k+1) = \overline{\mathcal{A}}\overline{\mathbf{x}}(k) + \overline{\mathbf{d}}(k), \tag{5.28}$$

where $\overline{\mathcal{A}} = \mathbf{T}^{-1}\mathcal{A}\mathbf{T}$, $\overline{\mathbf{d}}(k) = \mathbf{T}^{-1}\mathbf{d}(k)$. The structure of the matrix $\mathbf{T}$ is very special. The transformation matrix is the modal matrix of $\mathcal{A}$ (Definitions 2.6, and 2.8). It follows that the matrix $\mathcal{A}$ has to have n linearly independent eigenvectors (n eigenvalues different from zero), to build a non-singular matrix $\mathbf{T}$.

Proposition 5.4
The state-space representation of the discrete time compartmental system can be cast into diagonal form, if the matrix $\mathcal{A}$ has n eigenvalues different from zero, i.e. if the condition $\sum_{\substack{j=1 \\ j\neq i}}^{n} \beta_{ji}(k) \neq 1$ is satisfied.

Proposition 5.4 is based on Theorem 2.7.

For instance, let us consider the example network in Figure 6. If the parameters β have the values as can be seen in Table 4 at time step k, the state-space representation can be rewritten a into diagonal form.

β_{21}	β_{32}	β_{45}	β_{62}	β_{64}	β_{74}	β_{78}	β_{03}	β_{06}	β_{07}
0.85	0.2	0.95	0.5	0.3	0.6	0.72	0.75	0.5	0.6

Table 4: Leaving flow rates

As demand input we have $d_1 = 100$, d_5=200, and $d_8 = 300$ vehicles per time unit at the time step k. The matrix equation of this network is as follows,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \\ x_6(k+1) \\ x_7(k+1) \\ x_8(k+1) \end{bmatrix} = \begin{bmatrix} 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.85 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0.95 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.3 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0.4 & 0.72 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.28 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \\ x_7(k) \\ x_8(k) \end{bmatrix} + \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \\ 200 \\ 0 \\ 0 \\ 300 \end{bmatrix}. \tag{5.29}$$

The transformation matrix $\mathbf{T}$ is built by the eigenvectors, and its inverse are the following matrices,

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 0 & 0.0664 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2074 & -0.3760 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0.8296 & 0.7521 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4240 & -0.4775 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0251 & 0 \\ 1.0000 & 0 & -0.5185 & 0.5372 & 0 & -0.3180 & 0.3183 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & -0.8480 & 0.8186 & -0.9864 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1644 \end{bmatrix} \tag{5.30}$$

$$\mathbf{T}^{-1} = \begin{bmatrix} 6.0714 & 2.5000 & 0 & 0.7500 & 1.5833 & 1.0000 & 0 & 0 \\ -34.0000 & -4.0000 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 27.3237 & 4.8218 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15.0695 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.0000 & 5.4286 & 0 & 1.0000 & 6.0000 \\ 0 & 0 & 0 & 2.3585 & 44.8114 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 39.7912 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.0828 \end{bmatrix} \tag{5.31}$$

We obtain the diagonal state-space representation by,

$$\begin{bmatrix} \overline{\mathbf{x}}_1(k+1) \\ \overline{\mathbf{x}}_2(k+1) \\ \overline{\mathbf{x}}_3(k+1) \\ \overline{\mathbf{x}}_4(k+1) \\ \overline{\mathbf{x}}_5(k+1) \\ \overline{\mathbf{x}}_6(k+1) \\ \overline{\mathbf{x}}_7(k+1) \\ \overline{\mathbf{x}}_8(k+1) \end{bmatrix} = \begin{bmatrix} 0.1500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.3000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8000 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \overline{\mathbf{x}}_1(k) \\ \overline{\mathbf{x}}_2(k) \\ \overline{\mathbf{x}}_3(k) \\ \overline{\mathbf{x}}_4(k) \\ \overline{\mathbf{x}}_5(k) \\ \overline{\mathbf{x}}_6(k) \\ \overline{\mathbf{x}}_7(k) \\ \overline{\mathbf{x}}_8(k) \end{bmatrix} + \begin{bmatrix} 1506.9453 \\ 2732.3677 \\ -3400.0000 \\ 8962.2821 \\ 7958.2471 \\ 923.8095 \\ 2885.7143 \\ 1824.8288 \end{bmatrix} \tag{5.32}$$

It can be observed that the disturbance vector has turned into the vector $\overline{\mathbf{d}}$ after the eigenvalue decomposition. In the diagonal state-space each compartment has its disturbance, unlike the generic form (5.29). Decoupled subsystems are obtained with specifically weighted $\bar{\mathbf{d}}$-terms, like they were separate compartments.

The block diagram of the decoupled subsystems is depicted in Figure 15.

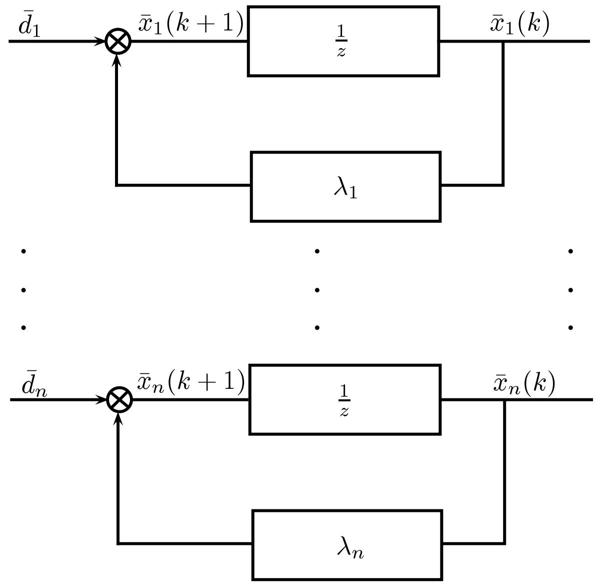

Figure 15: Block diagram of the diagonal representation

Actually, we used a well-known eigenvalue decomposition technique (Definition 2.6) to obtain (5.32). Naturally, since we applied a state-space tranformation, the physical meaning of the states will be changing. That is why they are denoted by overlined letters.

5.4 Sensitivity analysis

Let us assume that the system matrix changes because of a circumstance. It can be an interesting question how the eigenvalues of the system matrix will change? The sensitivity analysis can provide an answer. Based on the matrix norm (Definition 2.10) two parameters can be introduced depending on [18]: the *matrix condition number*, and the *eigenvalue condition number*, denoted by $\kappa(\mathbf{T})$ and $\kappa(\lambda, \mathcal{A})$, respectively. Note that in the matrix representation of the discrete time compartmental traffic system the control inputs are placed into the system matrix $\mathcal{A}$. Therefore, the change of control inputs causes changes in the system matrix. In the following subsection, the sensitivity of the matrix $\mathcal{A}$ is examined in terms of its eigenvalues.

5.4.1 Matrix condition number

This subsection highly depends on [18].

The nonsingular matrix $\mathcal{A}$ is examined from point of view sensitivity of its eigenvalues. The eigenvalue decomposition (Definition 2.6) can be given as,

$$\mathbf{D} = \mathbf{T}^{-1}\mathcal{A}\mathbf{T}. \tag{5.33}$$

Let $\delta\mathcal{A}$ denote a change in the matrix $\mathcal{A}$. Therefore, the eigenvalues have changes $\delta\mathbf{D}$, as well. Inserting these amounts into the previous formula, we come to:

$$\mathbf{D} + \delta\mathbf{D} = \mathbf{T}^{-1}(\mathcal{A} + \delta\mathcal{A})\mathbf{T}. \tag{5.34}$$

The changes of the eigenvalues are,

$$\delta\mathbf{D} = \mathbf{T}^{-1}\delta\mathcal{A}\mathbf{T}. \tag{5.35}$$

By taking matrix norms,

$$||\delta\mathbf{D}|| \leq ||\mathbf{T}^{-1}|| \cdot ||\mathbf{T}|| \cdot ||\delta\mathcal{A}|| = \kappa(\mathbf{T})||\delta\mathcal{A}||. \tag{5.36}$$

Definition 5.5 (Matrix condition number)
The matrix condition number, denoted by $\kappa(\mathbf{T})$ can be defined as,

$$\kappa(\mathbf{T}) = ||\mathbf{T}^{-1}|| \cdot ||\mathbf{T}||. \tag{5.37}$$

(Remark: the above equation is valid for each p-norm.) Actually, the sensitivity of the eigenvalues is computed based on the eigenvectors. The number $\kappa(\mathbf{T})$ concerns all of the eigenvalues, since it is a number. (On the other hand, it can be obtained as the product of two matrix norms, thus it cannot be nothing else than a number.)

5.4.2 Eigenvalue condition number

This subsection highly depends on [18].

To define the parameter appeared in the title of this section, both the left and the right eigenvectors (Definition 2.2) of the matrix $\mathcal{A}$ are required.

Definition 5.6 (Eigenvalue condition number)
The eigenvalue condition number can be defined as,

$$\kappa(\lambda, \mathcal{A}) = \frac{||\mathbf{y}|| \cdot ||\mathbf{v}||}{y^T v}, \tag{5.38}$$

where $\mathbf{y}$ and $\mathbf{v}$ denote the left, and the right eigenvectors of the matrix $\mathcal{A}$, respectively.

The deduction of this parameter can be found in [18]. There are distinct eigenvalue condition numbers for each eigenvalue, which can be stacked into a vector. Thus, the parameter $\kappa(\lambda, \mathcal{A})$ is a column vector in this approach.

It can be deduced that the matrix condition number gives an upper bound for the eigenvalue condition number, i.e.

$$\kappa(\lambda, \mathcal{A}) \leq \kappa(\mathbf{T}). \tag{5.39}$$

According to [1] the large value of the eigenvalue condition number denotes that the examined system matrix has eigenvalues close to each other. This parameter provides information how homogeneous are the leaving flow rates. Moreover, the greater entries in the vector $\kappa(\lambda, \mathcal{A})$ show which eigenvalues are more sensitive to the changes.

5.4.3 Examples

Matrix $\mathcal{A}$ of the example network in Figure 6 is examined again, but now from a point of view sensitivity of the eigenvalues. Three different set of β-s are applied for comparison. (The third row is the same as Table 2, and the second coincides the Table 4.)

Case	β_{21}	β_{32}	β_{45}	β_{62}	β_{64}	β_{74}	β_{78}	β_{03}	β_{06}	β_{07}
1	0.78	0.9	0.5	0.05	0.65	0.3	0.89	1	0.7	0.6
2	0.85	0.2	0.95	0.5	0.3	0.6	0.72	0.75	0.5	0.6
3	0.8	0.5	0.9	0.5	0.3	0.6	0.9	0.8	0.5	0.6

Table 5: Leaving flow rates at diagonalizability

Matrix $\mathcal{A}$ belongs to the first row is the following

$$\mathcal{A} = \begin{bmatrix} 0.22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.78 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.05 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.05 & 0 & 0.65 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0.4 & 0.89 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.11 \end{bmatrix}$$

The obtained results are stacked in Table 6.

Case	1	2	3
$\kappa(\lambda, \mathcal{A})$	$\begin{bmatrix} 19.5599 \\ 84.6626 \\ 66.3156 \\ 4.3568 \\ 5.1367 \\ 7.3384 \\ 5.4333 \\ 3,2278 \end{bmatrix}$	$\begin{bmatrix} 15.0694 \\ 27.7459 \\ 34.2491 \\ 44.8734 \\ 39.7912 \\ 6,8689 \\ 8.3946 \\ 6.0828 \end{bmatrix}$	$\begin{bmatrix} 4.5036 \cdot 10^{16} \\ 11.8427 \\ 4.5036 \cdot 10^{16} \\ 9.5595 \cdot 10^{16} \\ 9.5595 \cdot 10^{16} \\ 3.1102 \\ 3.7417 \\ 3.1623 \end{bmatrix}$
$\kappa(\mathbf{T})$	296.2547	158.6457	$3.0399 \cdot 10^{17}$

Table 6: Results of the sensitivity analysis

We can state that $\kappa(\mathbf{T})$ is greater than $\kappa(\lambda, \mathcal{A})$ in each cases. There are significant eigenvalue condition numbers among the computed sets. In Case 3 there are large condition numbers, since several eigenvalues are the same, i.e. the outflow proportion in compartments 1, and 3, resp. in compartments 4, and 5 are equal. In this case, the compartments with the same outflow are connected to each other. In the compartmental traffic system there are large condition numbers, associated to the compartments with the same outflow rate, connected to each other, or in case, if they are connected to the same compartment. This statement is explained more into details by the following examples. In Case 1 there are equal eigenvalues (belong to compartments 2, and 4), as well. But these two compartments are not connected to each other or to a same compartment, thus the condition numbers do not have great values.

In the sequel we perform sensitivity analysis of the network, while changing the control inputs. Recall the network example in Figure 6. The control inputs have been chosen as it can be seen in the second row in Table 5. It is well known that parameters β have a value between 0 and 1. The behaviour of the network changes by these parameters. What would the matrix condition number, and the eigenvalue condition number say us, if different β-s change their values? One could answer this property, if the control inputs are subjected to changes over the entire possible set, with a small increment. Thus, the corrensponding parameter β_{ji} has been changed between 0 and 1, while the rest of the parameters were kept constant.

As first case, let us examine the changing of β_{45}. This connection can be considered as a 1:1 connection, since at the sending compartment there is only one outgoing connection, and at the receiver compartment there is only one incoming connection. The values of $\kappa(\mathbf{T})$, and $\kappa(\lambda, \mathcal{A})$ are depicted in the following figures. By the latter parameter, each entry of the vector $\kappa(\lambda, \mathcal{A})$ is depicted in a detached diagram.

Based on Figure 16, it can be stated that the changing of β_{45} will affect the behaviour of four compartments. If we observe the figure accurately, we can appoint that these four compartments are the receiving compartment of the connection, and such compartments, which are connected to the receiving compartment. These compartments are called as the affected set of compartments, in the sequel. The compartments are excluded from the affected set are not influenced by the changing of observed connection. These statements can be verified in Figure 16.

The eigenvalue condition number belonging to the i-th compartment has a peak value, if β_{45} equals the amount $\sum_{\substack{j=1 \\ j\neq i}}^{n} \beta_{ji}$, and i is the element of the affected set of compartments. (Remark: the peak values have order of magnitude of 10^{15}, but in the figures they are cutted down for better visibility.) In case of the incoming compartment (in the present instance compartment 4) more peaks appear, in the same position as by the neighbouring compartments. Actually, if the β parameter of a 1:1 connection coincides with the outflow proportion of a compartment belonging to the affected set, two eigenvalues will be more senisitive.

Naturally, the matrix condition number has peaks in cases of the above mentioned values of β_{45}, as well (see Figure 17). It builds an upper bound to the eigenvalue condition number. Remark: the off-peak values are not zero, but they have quite neglectable value compared to the peaks.

According to my observations each 1:1 connection behaves the same way.

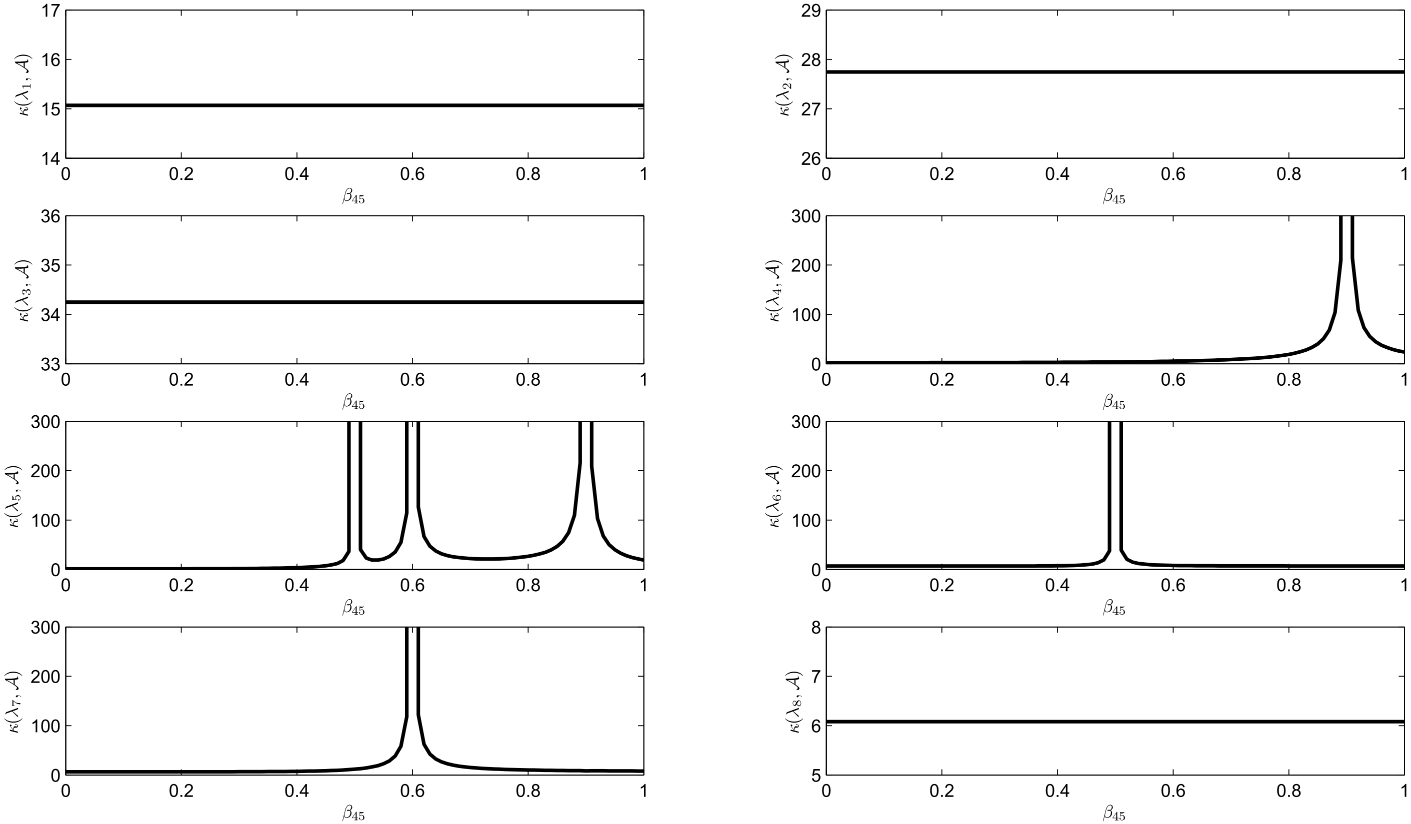

Figure 16: The changing of $\kappa(\lambda, \mathcal{A})$ caused by the changing of β_{45}

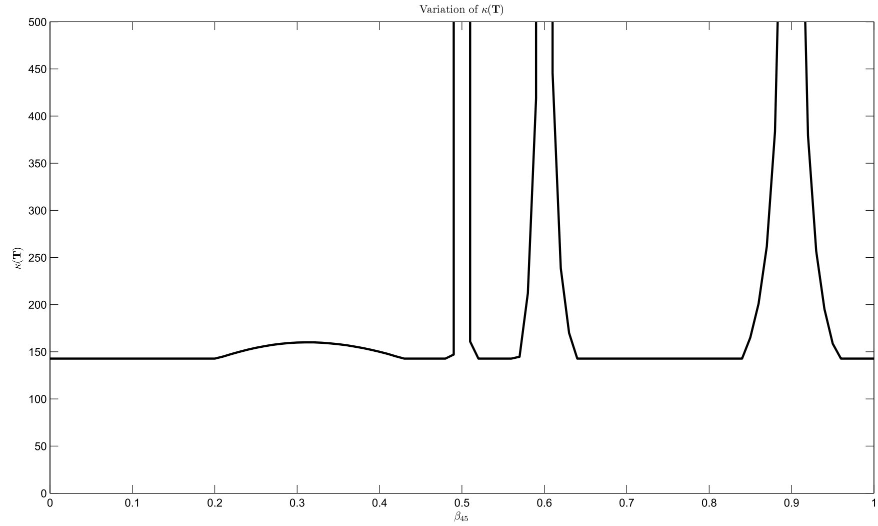

Figure 17: The changing of $\kappa(\mathbf{T})$ caused by the changing of β_{45}

How do the 1:n, n:1, and the n:n connections influence the behaviour of the system? Depending on a couple of example, it influences the same way. For instance, let us consider the connection 6-2 with β_{62}. The same experiment has been made, as by the β_{45}. The results can be seen in Figures 18, and 19.

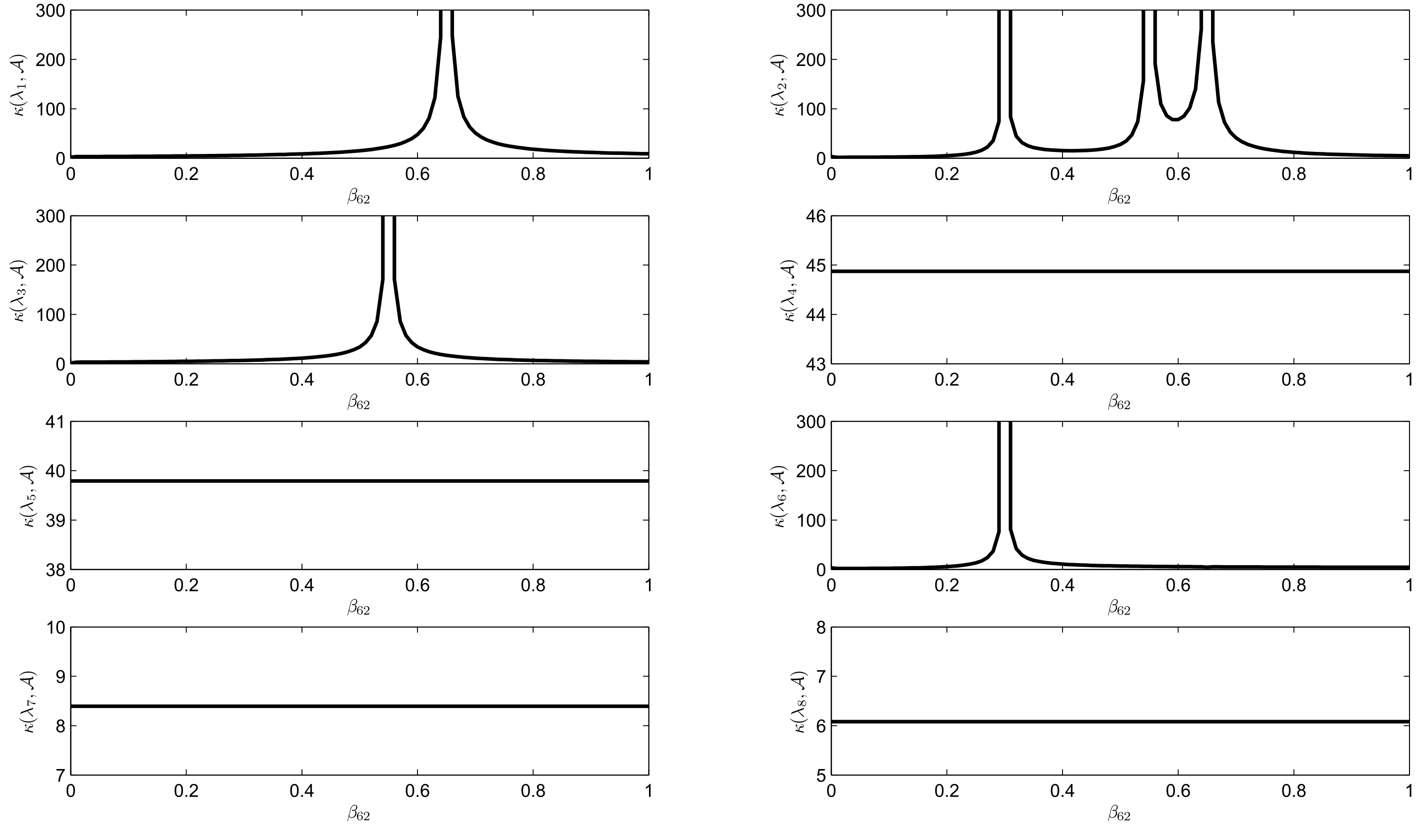

Figure 18: The changing of $\kappa(\lambda, \mathcal{A})$ caused by the changing of β_{62}

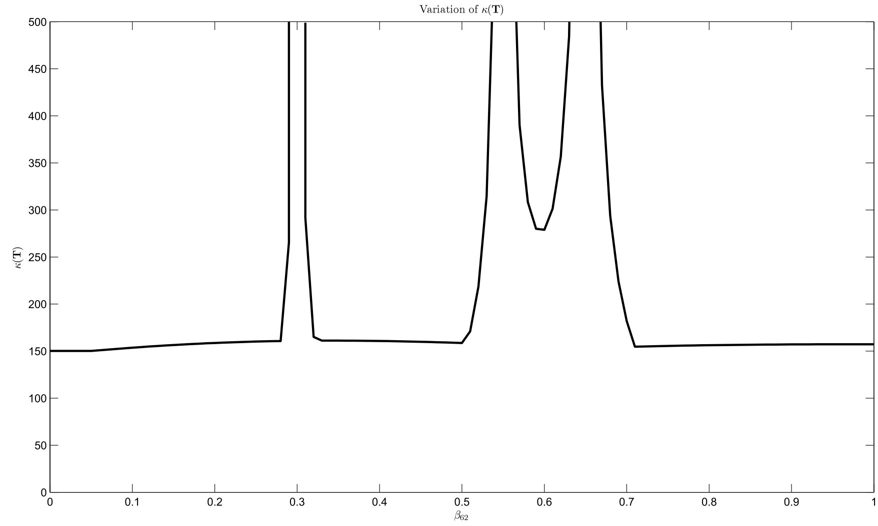

Figure 19: The changing of $\kappa(\mathbf{T})$ caused by the changing of β_{62}

The obtained diagrams are very similar to the previous case, but there is an important difference. The affected set of compartments consists of the sender compartment of the examined connection, as well as the compartments connected to the sender compartment. Apart from this, the changes behave in the same way as by a 1:1 connection.

In case of the outflows to the environment (β_{0i}), the affected set of compartments contains other compartments. All of the compartments, from which the examined outflow to the environment is accessible, belong to the affected set. The results can be seen in Figures 20, and 21.

There is a peak in the changing of the eigenvalue condition number at the i-th compartment, if the changing β_{06} has the same value, as the sum $\sum_{\substack{j=1 \\ j\neq i}}^{n} \beta_{ji}$. The eigenvalue condition number belonging to the compartment 6 has three peaks, on the locations, where the other elements of the affected set.

As summary, we can say that, the condition numbers are connected with the structure of the graph in a certain sense. They have peak values, if connected compartments have the same leaving flow rate. The independent compartments with the same outflow do not cause such changing in the condition numbers. A low value of $\kappa(\lambda, \mathcal{A})$, or $\kappa(\mathbf{T})$ indicates a heterogeneous eigenvalue structure, i.e. the leaving flow rates have distinct values. The theory of the presented sensitivity analysis can be related to the unicity of the solution of difference equation.

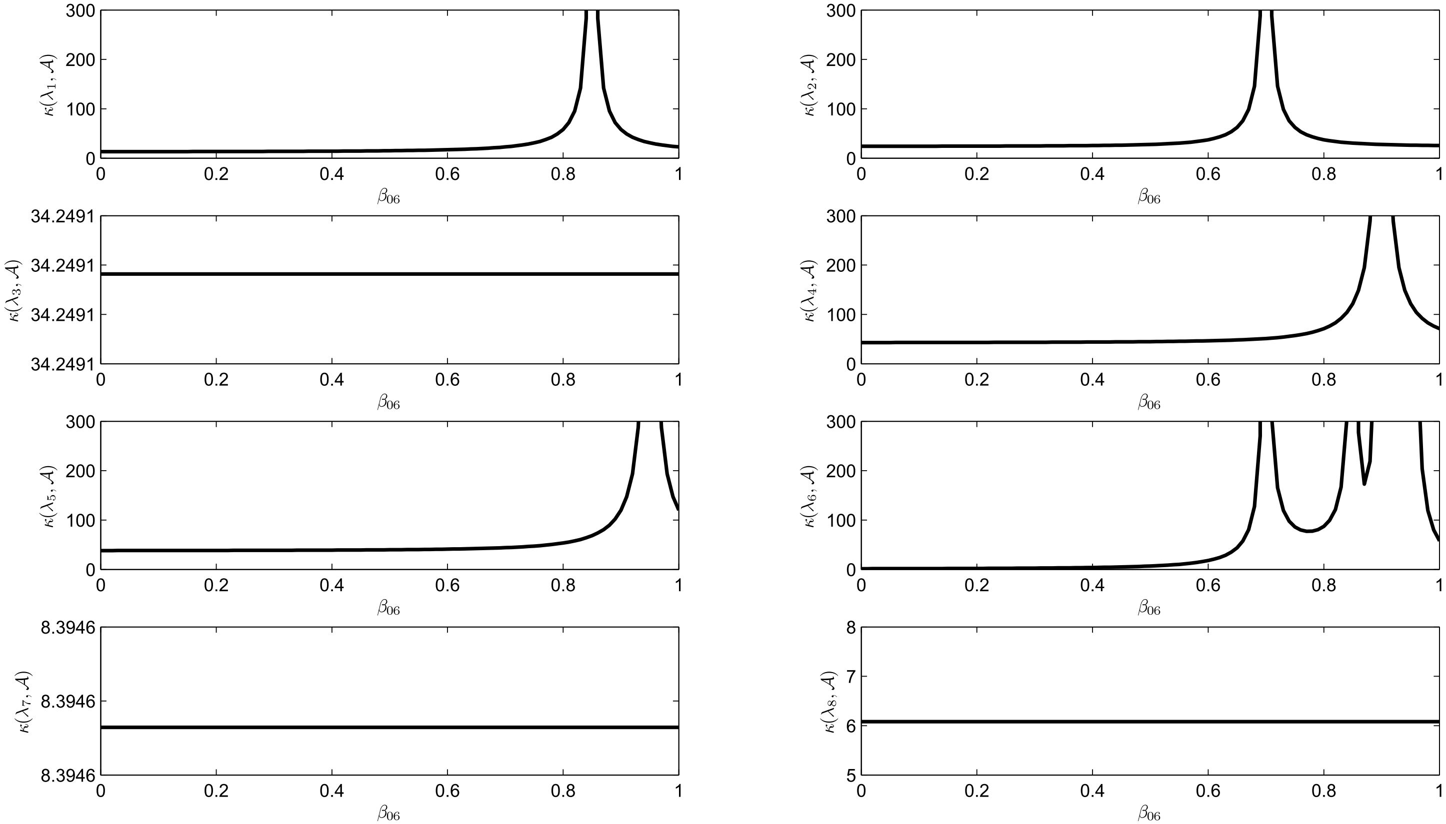

Figure 20: The changing of $\kappa(\lambda, \mathcal{A})$ caused by the changing of β_{06}

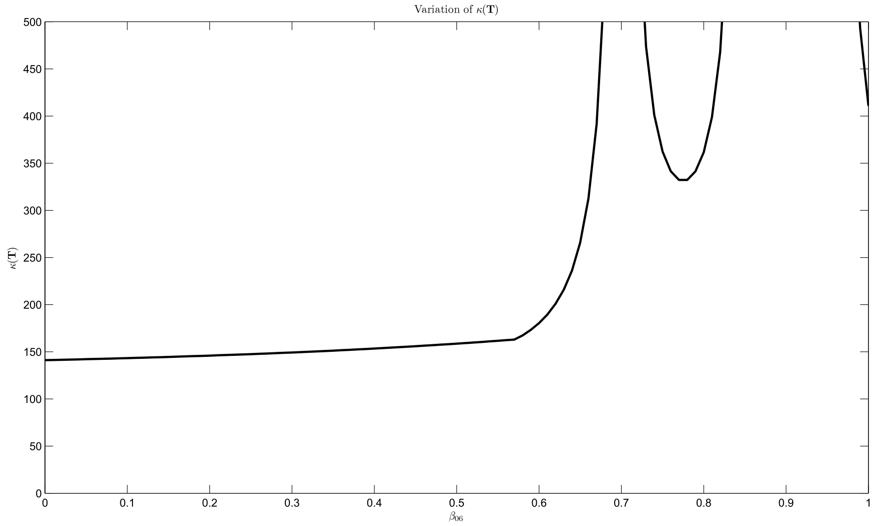

Figure 21: The changing of $\kappa(\mathbf{T})$ caused by the changing of β_{06}

6 Further research directions

Some topics are proposed in the sequel, which have to be examined more detailed. The introduction of green times into the model, the consideration of the uncertainties in the compartmental traffic model are skimmed. A review of the possible way of controls is presented, as well.

6.1 Green times

To influence traffic flow in urban area, we often use traffic lights. In traffic modelling it is a reasonable question, how we compute the green time, as the effective control input. We have to connect the outflow of the road links with the green time in a suitable way. The SF model applies a leaving capacity for each link (denoted by s), which simply says how many vehicles can pass the stopline during a time unit. In the easiest case this capacity is chosen as a constant value (in most cases 0.5 vehicle in each second). Unfortunately, exactly this computed number of vehicles from a given link will flow out in a few cases. The traffic conditions and the driver behaviour affect the outflow capacity. Thus, the outflow will change around the ideal value, and it follows from the previous statements that not so many vehicles will leave the intersection, as it has been computed previously. Actually, we can model the outflow with uncertainty, such as the sum of a nominal outflow parameter, and a changing of the outflow: $h_i(k) = h_i^n(k) + \delta h_i(k)$. By this amount we can obtain for the green time:

$$u_i(k) = \frac{h_i^n(k) + \Delta h_i(k)}{s_i}. \tag{6.1}$$

The above equation can be rephrased as,

$$u_i(k) = \frac{h_i^n(k)}{s_i} + \frac{\Delta h_i(k)}{s_i}. \tag{6.2}$$

The green time can be given as a sum of a nominal and an uncertain part, such as

$$u_i(k) = u_i^n(k) + \Delta u_i(k). \tag{6.3}$$

Naturally, the control of the urban traffic is always a constrained control task. From [2, 3, 10] it is deducible that the following constraints influence the green time:

1.
$$u_{min} \leq u_i \leq u_{max}, \quad \forall i \tag{6.4}$$

2.
$$\sum_{i=1}^{n} u_i = T_c - \sum t_{ig}, \tag{6.5}$$

   where t_{ig} denotes the intergreen time.

If we consider the outflow with uncertainty, we can reformulate the above constraints into the following form by using the formula: (6.3)

1.
$$u_{i_{min}}^n(k) + \Delta u_{i_{min}}(k) \leq u_i(k) \leq u_{i_{max}}^n(k) + \Delta u_{i_{max}}(k), \tag{6.6}$$

2.
$$\sum_{i=1}^{n} u_i^n(k) + \Delta u_i(k) = T_c - \sum t_{ig}. \tag{6.7}$$

We have obtained uncertain constraints by this step, which cannot be used well in solving control problems.

In case of the compartmental traffic system it can be an obvious solution to use a leaving capacity similar to the SF approach. We can choose the green time of the i-th compartment similarly to the SF model as,

$$u_i(k) = \frac{h_i(k)}{s^*} = \frac{\sum_{\substack{l=1 \\ l \neq i}}^{n} \beta_{li}(k) x_i(k)}{s^*}, \tag{6.8}$$

where s^* denotes a leaving capacity. Constraints to the green time are required in the compartmental approach, as well. By an uncertain outflow we would obtain very similar uncertain constraints, as by the SF model.

But do we need this leaving capacity? Let us consider another example by trying to detach the outflow and the green time. Initially it is assumed that the number of outflowing vehicles is measured at each stop line, and the order of the phases is fixed. The control algorithm computes only the proportions β by keeping the constraints (4.2), and (4.3). Thus, it is computed how many vehicles have to flow out from the different directions in the next cycle. The traffic controller does not compute green times in advance. It only gives green signal, and counts how many vehicles have already flowed out. When this amount equals to the previously appointed number, the traffic sign turns into amber, and red, and follows the next phase.

By introducing fixed cycle time to the signal plan, the previous approach can cause serious problems. Because of the uncertainties there can arise such cases, when the first n phases need so much green time that the actual cycle reaches its end before all phases would take place. The application of the fixed cycle time is not definitely reqiured, but if the cycle time is subject to change, the signal plan of the junction will be more unpredictable.

6.2 Uncertainties

As it was discussed in the previous section, the outflow through the stop line cannot be described by an exact value. This information can be built into the compartmental approach by the introduction of the nominal β and the uncertain β.

Define $\beta^n_{ij}(k)$ as a nominal outflow rate. If each driver and vehicle would be the same, this parameter would be applicable independently. Since, the participants in the traffic process have different properties, another term, the changing $(\Delta\beta_{ij}(k))$ is introduced. Thus, the entire $\beta_{ij}(k)$ is generated by the sum of the above two amounts as,

$$\beta_{ij}(k) = \beta^n_{ij}(k) + \Delta\beta_{ij}(k). \tag{6.9}$$

Thus, we obtain for the matrix $\mathcal{A}$ the same structure, as by the case without uncertainty, but the nonzero entries are subject to change as it can be seen in equation (6.10). Matrix $\mathcal{A}$ can be dissociated into two parts: to a nominal and an uncertain part with state additive structure.

6.3 Ideas for control

Finding a controller to a nonlinear system is not a trivial case. Moreover, in traffic applications parameters always appear, which have to be minimized or maximized during the control. A typical choice can be the minimizing of the total time spent parameter (defined in [16], as well). Linear quadratic (LQ) control, and the various types of the model predictive control (MPC) are able to handle minimization or maximiziation problems during the computation of control inputs. However, each methodology can be acceptable, which can handle constraints.

Similarly to quite all physical processes, the traffic flow goes between constraints, which fact has to be considered in the controller. For instance, the road network has capacity bound related to the throughput. Most of the traffic lights cannot give a green signal with arbitrary length. The MPC techniques can handle such restrictive conditions in the state resp. the control input.

The connection between the green time and the outflow of the links becomes very important at this point. If the uncertainty of the outflows appears in the green times (for instance by the terms $g_1(k) = g_1^n(k) + \Delta g_1(k)$) this uncertainty will be adapted into the constraints, as it can be seen in (6.6). The control task with uncertain constraints will not able to give optimal solution, since the uncertain part can change from step to step. It can be estimated depending on historical data, but this is not the fanciest solution.

$$
\mathcal{A} = \begin{bmatrix}
1 - \sum\limits_{\substack{j=1\\ j\neq 1}}^{n} \beta_{j1}^{n}(k) - \sum\limits_{\substack{j=1\\ j\neq 1}}^{n} \Delta\beta_{j1}(k) - \beta_{01}^{n}(k) - \Delta\beta_{01}(k) & \cdots & \beta_{1i}^{n}(k) + \Delta\beta_{1i}(k) & \cdots & \beta_{1n}^{n}(k) + \Delta\beta_{1n}(k) \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\beta_{i1}^{n}(k) + \Delta\beta_{i1} & \cdots & 1 - \sum\limits_{\substack{j=1\\ j\neq i}}^{n} \beta_{ji}^{n}(k) - \sum\limits_{\substack{j=1\\ j\neq i}}^{n} \Delta\beta_{ji}(k) - \beta_{0i}^{n}(k) - \Delta\beta_{0i}(k) & \cdots & \beta_{in}^{n}(k) + \Delta\beta_{in}(k) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{n1}^{n}(k) + \Delta\beta_{n1} & \cdots & \beta_{ni}^{n}(k) + \Delta\beta_{ni} & \cdots & 1 - \sum\limits_{\substack{j=1\\ j\neq n}}^{n} \beta_{jn}^{n}(k) - \sum\limits_{\substack{j=1\\ j\neq n}}^{n} \Delta\beta_{jn}(k) - \beta_{0n}^{n}(k) - \Delta\beta_{jn}(k)
\end{bmatrix} \tag{6.10}
$$

$$
\mathcal{A} = \mathcal{A}^{n} - \Delta\mathcal{A} = \begin{bmatrix}
1 - \sum\limits_{\substack{j=1\\ j\neq 1}}^{n} \beta_{j1}^{n}(k) - \beta_{01}^{n}(k) & \cdots & \beta_{1i}^{n}(k) & \cdots & \beta_{1n}^{n}(k) \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\beta_{i1}^{n}(k) & \cdots & 1 - \sum\limits_{\substack{j=1\\ j\neq i}}^{n} \beta_{ji}^{n}(k) - \beta_{0i}^{n}(k) & \cdots & \beta_{in}^{n}(k) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{n1}^{n}(k) & \cdots & \beta_{ni}^{n}(k) & \cdots & 1 - \sum\limits_{\substack{j=1\\ j\neq n}}^{n} \beta_{jn}^{n}(k) - \beta_{0n}^{n}(k)
\end{bmatrix} - \tag{6.11}
$$

$$
- \begin{bmatrix}
\sum\limits_{\substack{j=1\\ j\neq 1}}^{n} \Delta\beta_{j1}(k) + \Delta\beta_{01}(k) & \cdots & \Delta\beta_{1i}(k) & \cdots & \Delta\beta_{1n}(k) \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\Delta\beta_{i1}(k) & \cdots & \sum\limits_{\substack{j=1\\ j\neq i}} \Delta\beta_{ji}(k) + \Delta\beta_{0i}(k) & \cdots & \Delta\beta_{in}(k) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Delta\beta_{n1}(k) & \cdots & \Delta\beta_{ni}(k) & \cdots & \sum\limits_{\substack{j=1\\ j\neq n}}^{n} \Delta\beta_{jn}(k) + \Delta\beta_{0n}(k)
\end{bmatrix} \tag{6.12}
$$

7 Summary and conclusion

We conclude the achieved results of the thesis project in the sequel.

In this thesis a novel approach for modeling urban traffic has been proposed. This new approach is the application of the well-known compartmental systems for modeling traffic flow in urban environment. Compartmental systems always hold the mass-conservation law, moreover they are always nonnegative. These two properties can be related to the traffic flow, as well. Mass-conservation appears as vehicle-conservation. The positivity is obviously required, since negative number of vehicles does not exists.

We have found an appropriate parameter - the leaving flow rate - to model the outflow of the vehicles in the intersections, moreover it can be used for control input. The leaving flow rate with its constraints ensures the fulfilment of the vehicle-conservation law, and the positivity. This parameter has led to a bilinear system structure, where the control inputs are involved the system matrix. The matrix representation of the compartmental traffic system always carries information about the network topology, unlike the SF model, in which the system matrix is a simple unit matrix.

The model analysis has provided information about the stability of the compartmental traffic system. We found that the system can be described as a stable one with constant β-s. With changing leaving flow rates only local stability can be appointed. Linear Lyapunov-function can be chosen in case of the compartmental systems. The number of vehicles present in the network can be considered as Lyapunov-function. Local controllability can be analised in this problem formulation. We have been examined the possibility of the decoupling of the system, which can be reduced to conditions to the control inputs. Sensitivity analysis of the compartmental traffic network is possible by the condition numbers of the system matrix. This field can be further examined in terms of the unicity of the solution of difference equation.

The conversion of the leaving flow rate to green time is not a trivial transformation, if we would like to avoid the application of the leaving capacity notion of the SF model. The uncertainties of the outflow can be built in the model by defining nominal and uncertain part of the system matrix. Further analysis is required to provide a comprehensive model for control tasks. The design process will be complete with an optimal control design.

The compartmental traffic model has an important advantage compared to the SF model. The system matrix $\mathcal{A}$ gives informations about the network topology, unlike the identity matrix of the SF model. Moreover, at the model analysis we can connect the statements of system theory to parts of the physical process. This step is quite unpossible at the SF model.

As final summary we can state that the compartmental traffic model is appropriate for model analysis. Control tasks depending on this model can be solved in a difficult way.

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