BME KJIT
Budapest University of Technology and Economics
Faculty of Transportation Engineering and Vehicle Engineering
Department of Control for Transportation and Vehicle Systems

Automotive Environment Sensing

04 – Linear estimation, Kalman filter

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Linear estimation

Situation

- We want to know the value of some quantity x
- We have two sensors with different precisions
- Based on the two measurements $(z_1,\ z_2)$ give a linear estimation of x
  - The measurements are corrupted by zero mean Gaussian noise

$$\mathrm{E}[x] = m \qquad \mathrm{E}[z_1] = m \qquad \mathrm{E}[z_1] = m$$

  - The noise STDs are σ_1 and σ_2

Linear estimation: $\hat{x} = a_1 z_1 + a_2 z_2$

Should we use both z or just the one with the smaller σ?

Linear estimation

Requirements for a good estimation

- Unbiased: the expectation of the estimated value equals the real value

$$\mathrm{E}[\hat{x}] = a_1\mathrm{E}[z_1] + a_2\mathrm{E}[z_2]$$

$$m = a_1 m + a_2 m$$

$$\boxed{1 = a_1 + a_2}$$

$$\hat{x} = a_1 z_1 + a_2 z_2 \qquad \boxed{\hat{x} = z_1 + a_2(z_2 - z_1)}$$

Linear estimation

Requirements for a good estimation

- Minimum variance: the variance of the estimation should be minimal

$$\mathrm{Var}(\hat{x}) = \mathrm{E}[(\hat{x} - E[\hat{x}])^2] =$$

$$= \mathrm{E}[(a_1(z_1 - m) + a_2(z_2 - m))^2]$$

$$= a_1^2 \mathrm{E}[(z_1 - m)^2] + a_2^2 \mathrm{E}[(z_2 - m)^2] + 0 =$$

$$= \boxed{a_1^2\sigma_1^2 + a_2^2\sigma_2^2}$$

Linear estimation

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$$a_1^2\sigma_1^2+a_2^2\sigma_2^2 = a_1^2\sigma_1^2+(1-a_1)^2\sigma_2^2$$

Set the derivative of the variance with respect to a_1 and a_2 equal to zero:

$$0 = \frac{\partial \mathrm{Var}(\hat{x})}{\partial a_1} = 2a_1\sigma_1^2 - 2(1-a_1)\sigma_2^2$$

$$\boxed{a_1 = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}} \quad \boxed{a_2 = \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}}$$

$$\hat{x} = a_1 z_1 + a_2 z_2$$

We give greater weight to the measurement with smaller noise

Linear estimation

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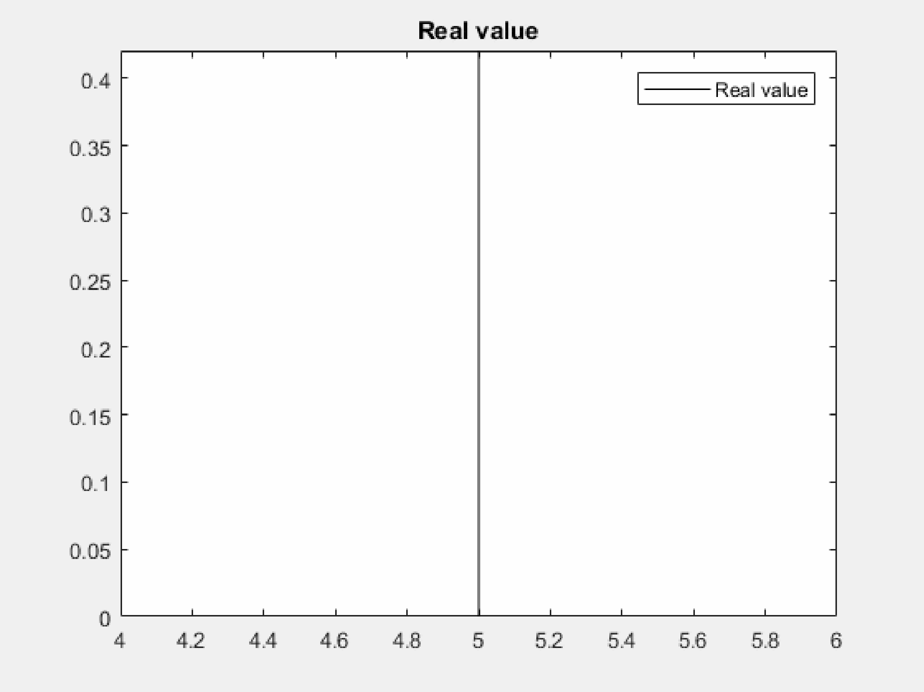
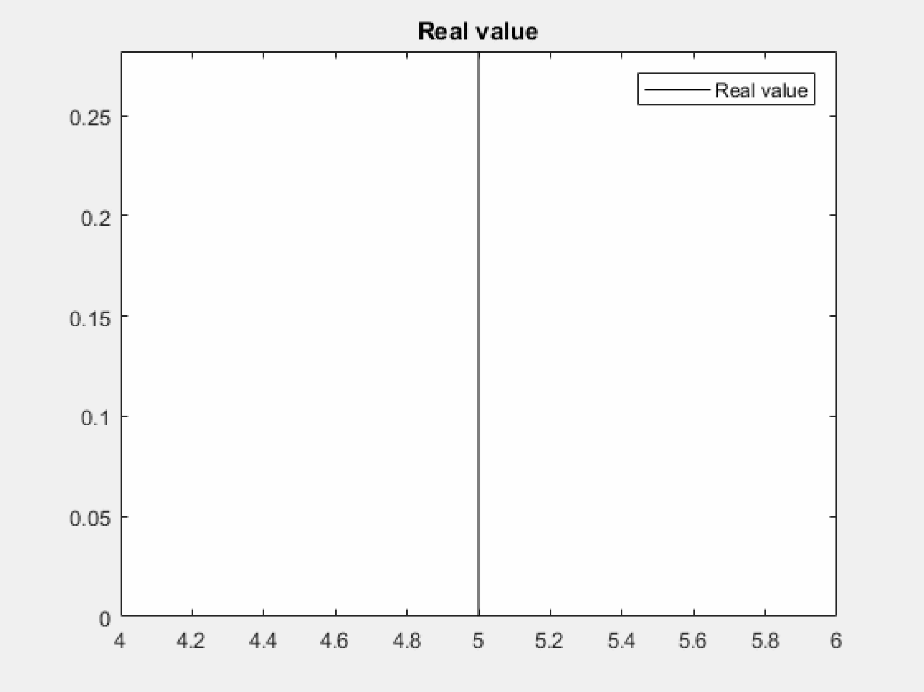
- Sensors with different precision



- Sensors with same precision

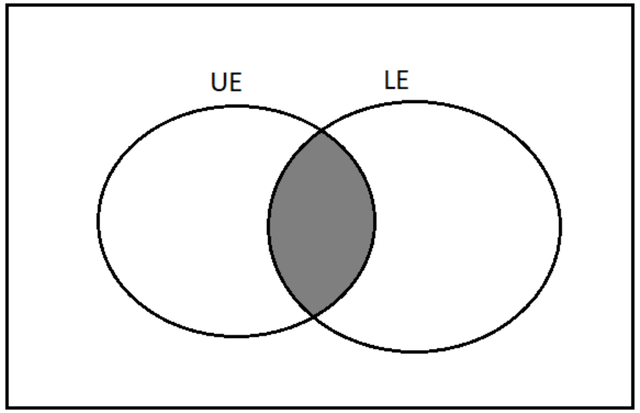


Linear estimation

Does the estimated value $\hat{x}$ have smaller variance than either σ_1^2 or σ_2^2?
With a_1 and a_2 substituted we have

$$\mathrm{Var}(\hat{x}) = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

Which is smaller than either σ_1^2 or σ_2^2
(Analogous to parallel resistors)
BLUE: Best Linear Unbiased Estimator
MVUE: Minimum Variance Unbiased Estimator



If the noise is Gaussian, then the BLUE is also minimum variance.

Linear estimation – with model

- What if, instead of two measurements we have one measurement and one predicted value based on some model?
- Let us change the notations:
  - $z_1 \to x_0$ is the prediction
  - $z_2 \to z$ is the measurement
- Simple example
  - Nearly constant velocity motion in a straight line
  - We are looking for the position
  - We can measure the position with error

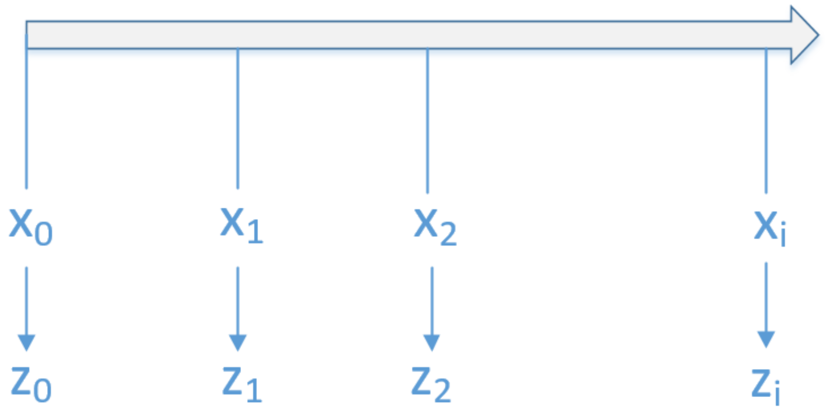
Linear estimation – with model

Discrete time nearly constant velocity motion in one dimension:

$$x_{k+1} = x_k + Tv + w_k$$

- T: timestep
- v: constant speed
- w: noise acting on the motion
- v: noise on the measurements

- Noisy measurements: $z_k = x_k + v_k$
- Estimated position: $\hat{x}$



Starting from a random position around zero

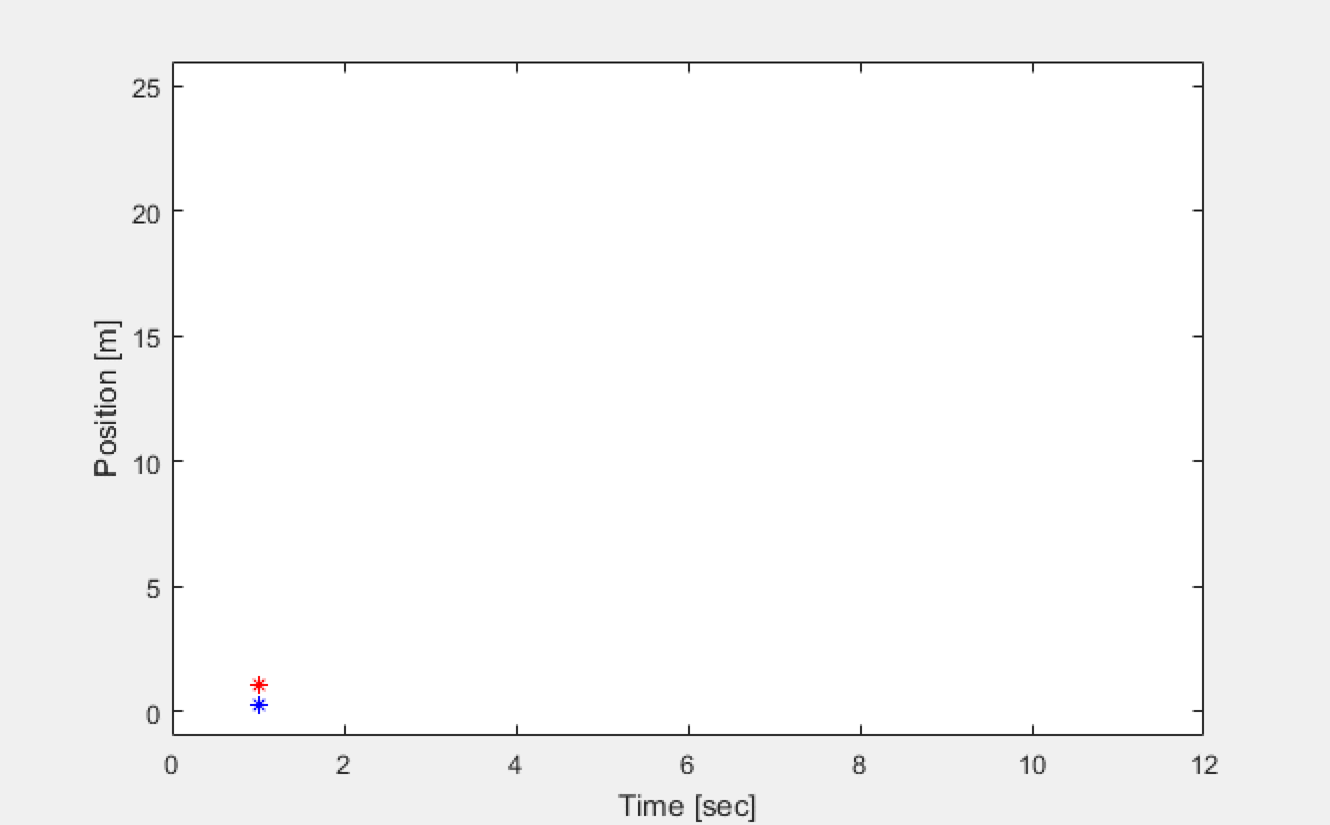
$$x_0 = 0 + w_0$$

Initialize the estimation with the first measurement

$$z_0 = x_0 + v_0$$

$$\hat{x}_0 = z_0$$

Linear estimation – with model



State observer – Luenberger observer

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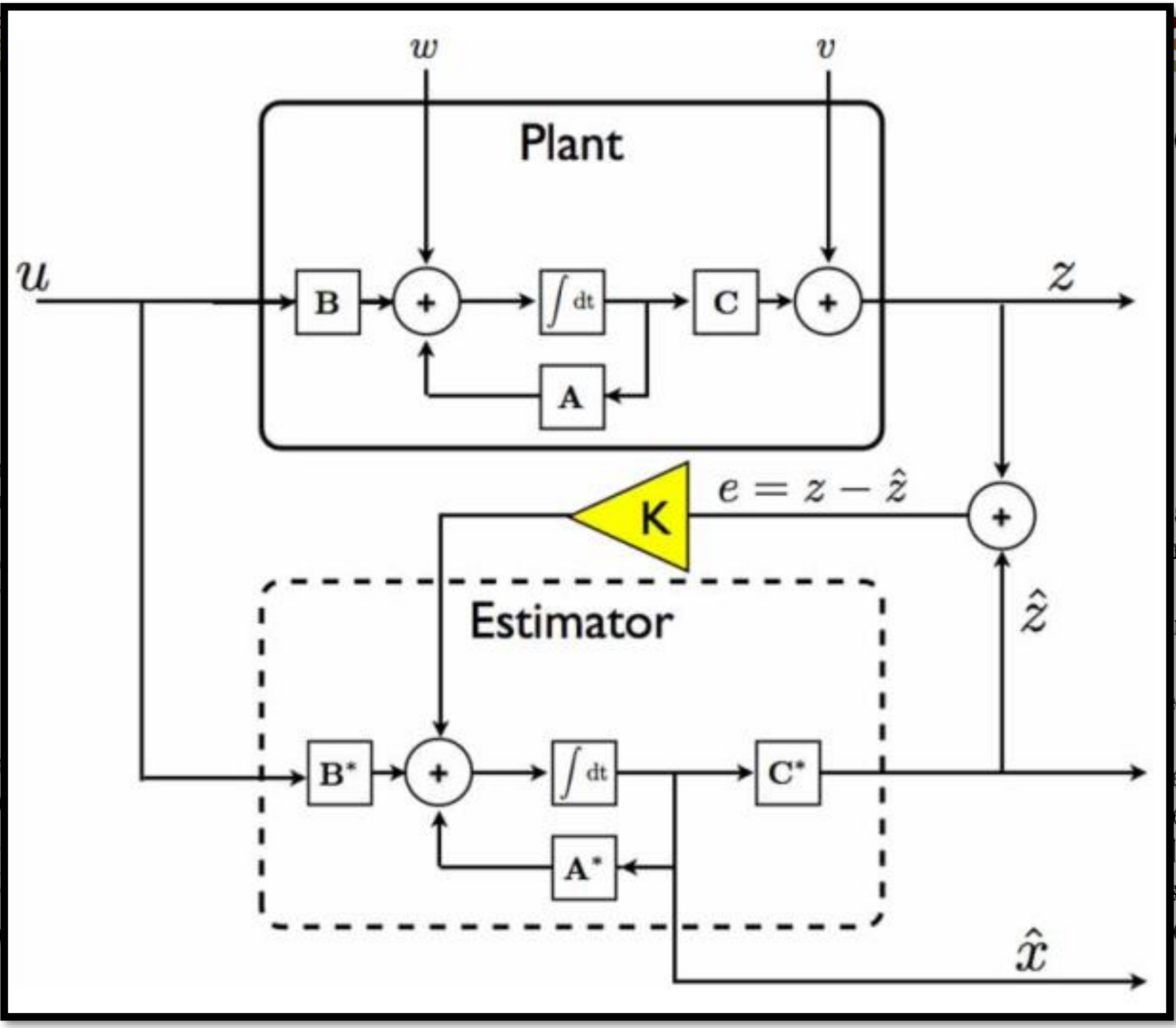


LO vs KF

$$x_{k+1} = \mathbf{\Phi}_k x_k + \mathbf{\Delta}_k u_k + \mathbf{\Gamma}_k w_k$$
$$z_k = \mathbf{H}_k x_k + v_k$$

$$\hat{x}_{k+1} = \mathbf{\Phi}_k \hat{x}_k + \mathbf{\Delta}_k u_k + \mathbf{K}[z_k - \hat{z}_k]$$
$$\hat{z}_k = \mathbf{H}_k \hat{x}_k .$$

- Luenberger observer has low performance when noise is introduced to the system
- Kalman filter: Linear Quadratic Gaussian Estimator
  - Linear model
  - Quadratic cost function
  - Gaussian noise

Why quadratic cost?

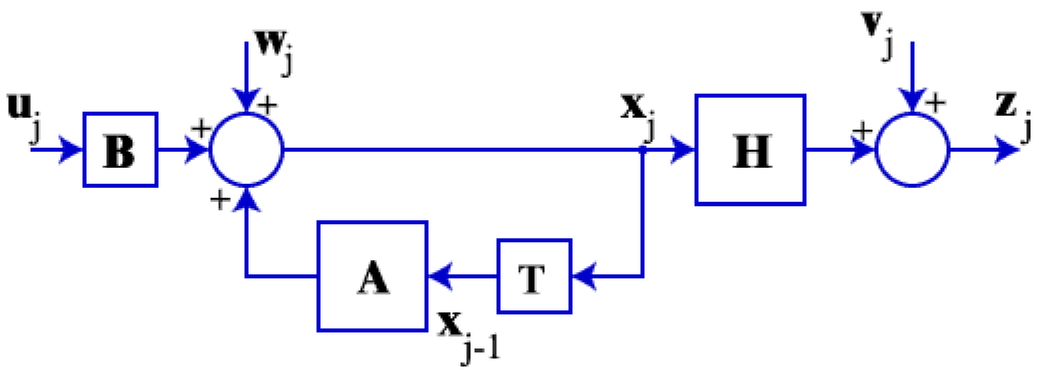
- Positive definite: $x^\top A x > 0$ for every non-zero x
- Symmetric positive definite (SPD) matrices have nice features:
  - Positive eigenvalues
  - A quadratic form is convex, if A is SPD
  - $Q(x) = \frac{1}{2}x^\top A x - b^\top x + c$
  - $\min_x \left(\frac{1}{2}x^\top A x - b^\top x + c\right)$ and $Ax = b$ has the same solution

RMS

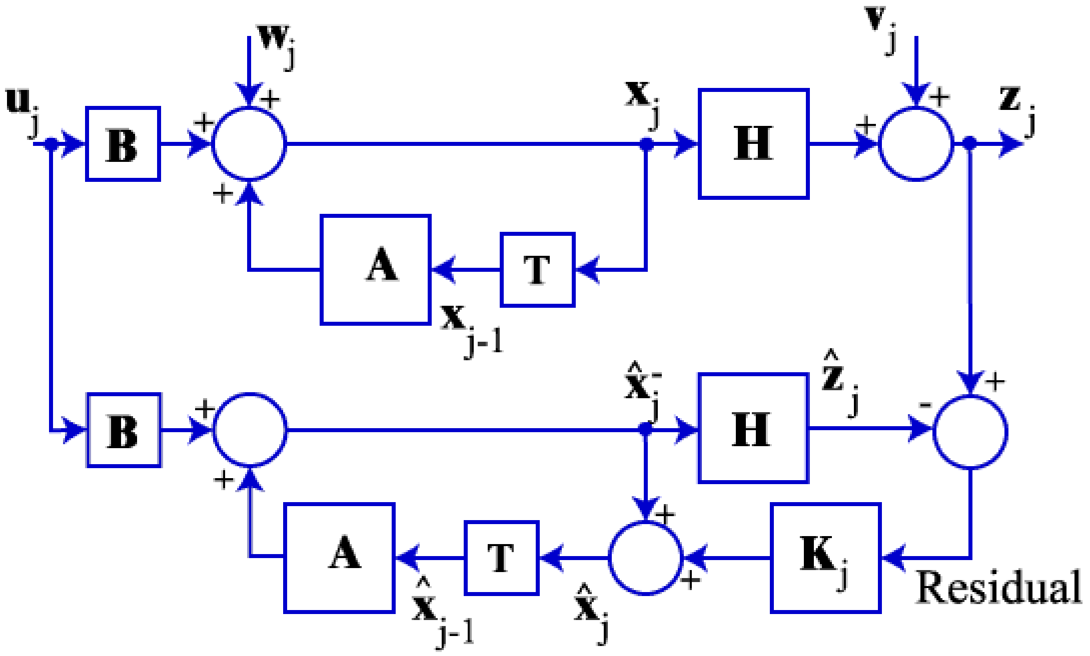
- x is normally distributed random vector: $x \sim \mathcal{N}(\mu, \Sigma)$
- If x describes a signal what is the expectation of the carried power?

$$\mathrm{E}[\|x\|_2^2] = \|\mu\|_2^2 + \mathrm{tr}(\Sigma)$$

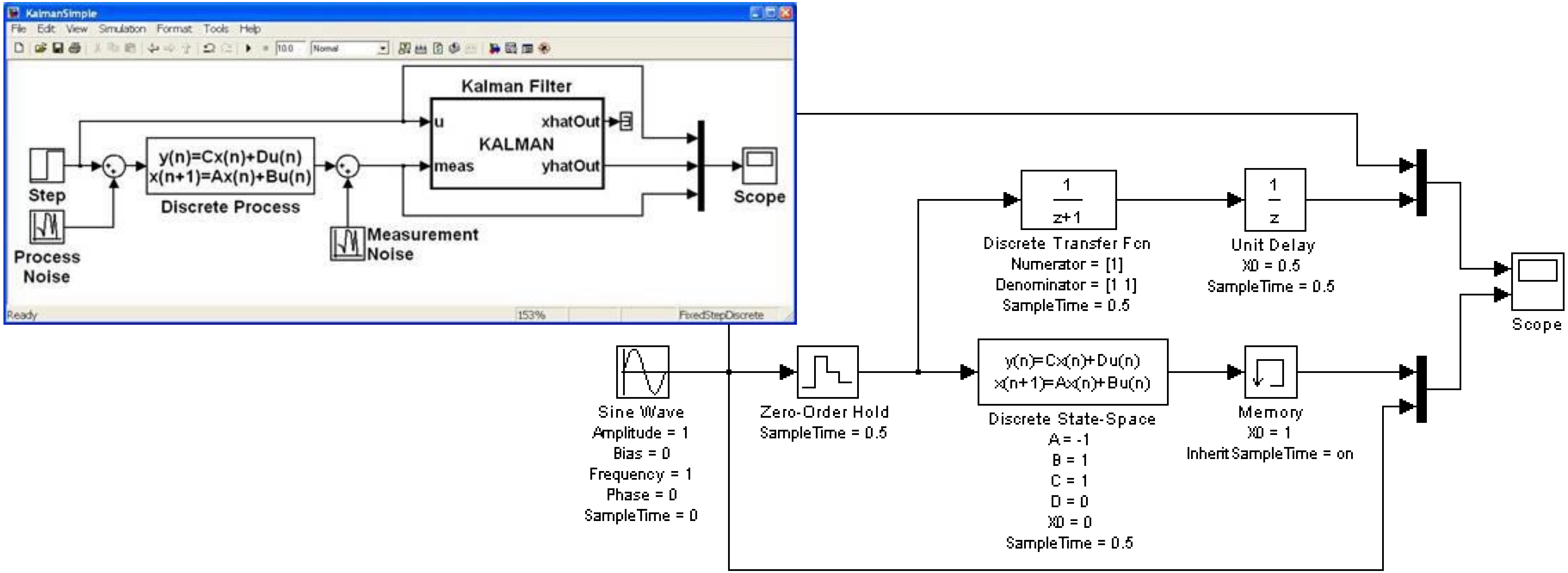
Plant block diagram

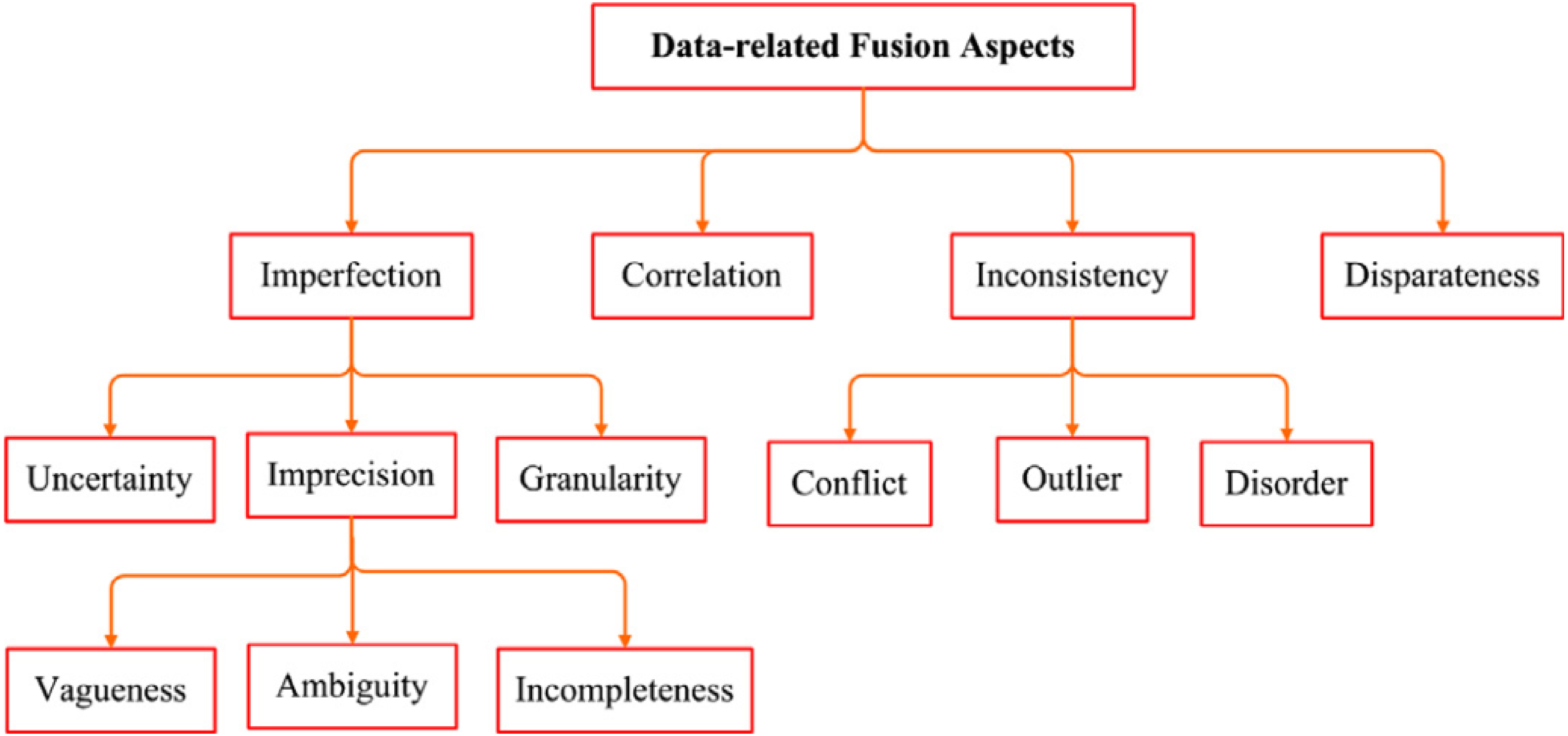


Plant and observer



Simulink Discrete State Space




Data-related Fusion Aspects
Imperfection
Correlation
Inconsistency
Disparateness
Uncertainty
Imprecision
Granularity
Conflict
Outlier
Disorder
Vagueness
Ambiguity
Incompleteness

Data fusion

- Probability theory (Bayesian inference)
- Dempster-Shafer theory (evidential belief reasoning)
- Fuzzy set theory (fuzzy reasoning)
- Possibility theory
- Rough set theory
- Random set theory

- Hybrid fusion
  - Fuzzy+DS
  - Fuzzy+Rough set

Method	Addressed data imperfection
Probabilistic	Uncertainty
Dempster-Shafer	Uncertainty and ambiguity
Fuzzy	Vagueness
Possibilistic	Incompleteness
Rough set	Ambiguity
Random set	Imperfection

Terminology

- Detection: knowing the presence of an object
- Tracking: Maintaining the state of a moving object over time using remote sensor measurements. In case of multi-target tracking the object has to be identified too